On Matrix-Valued Stieltjes Functions with an Emphasis on Particular Subclasses

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Dedicated to our friend and colleague Albrecht Böttcher on the occasion of his 60th birthday

The paper deals with particular classes of $q \times q$ matrix-valued functions which are holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$, where α is an arbitrary real number. These classes are generalizations of classes of holomorphic complex-valued functions studied by Kats and Krein [17] and by Krein and Nudelman [19]. The functions are closely related to truncated matricial Stieltjes problems on the interval $[\alpha, +\infty)$. Characterizations of these classes via integral representations are presented. Particular emphasis is placed on the discussion of the Moore-Penrose inverse of these matrix-valued functions.

1. Introduction

In their papers [11,13,14], the authors developed a simultaneous approach to the even and odd truncated matricial Hamburger moment problems. This approach was based on three cornerstones. One of them, namely the paper [11] is devoted to several functiontheoretical aspects concerning special subclasses of matrix-valued Herglotz-Nevanlinna functions. Now we are going to work out a similar simultaneous approach to the even and odd truncated matricial Stieltjes moment problems. Our approach is again subdivided into three steps. This paper is concerned with the first step which is aimed at a closer analysis of several classes of holomorphic matrix-valued functions in the complex plane which turn out to be closely related to Stieltjes type matrix moment problems. In the scalar case, the corresponding classes were carefully studied by Kats/Krein [17] and Krein/Nudelman [19, Appendix]. What concerns the treatment of several matricial and operatorial generalizations, we refer the reader to the monograph Arlinskii/Belyi/Tsekanovskii [1] and the references therein.

1. Introduction

In order to give a precise formulation of the matricial moment problem standing in the background of our investigations, we introduce some notation. Throughout this paper, let p and q be positive integers. Let $\mathbb{C}$ and $\mathbb{R}$ be the set of all complex numbers and the set of all real numbers, respectively. Furthermore, let $\mathbb{N}_0$ and $\mathbb{N}$ be the set of all non-negative integers and the set of all positive integers, respectively. Further, for every choice of $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$, let $\mathbb{Z}_{\alpha,\beta}$ be the set of all integers k for which $\alpha \leq k \leq \beta$ holds. If $\mathcal{X}$ is a non-empty set, then let $\mathcal{X}^{p \times q}$ be the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^p$ is short for $\mathcal{X}^{p \times 1}$. The notations $\mathbb{C}_{\mathrm{H}}^{q \times q}$, $\mathbb{C}_{\geq}^{q \times q}$, and $\mathbb{C}_{>}^{q \times q}$ stand for the subsets of $\mathbb{C}^{q \times q}$ which are formed by the sets of Hermitian, non-negative Hermitian, and positive Hermitian matrices, respectively. If $(\Omega, \mathfrak{A})$ is a measurable space, then each countably additive mapping whose domain is $\mathfrak{A}$ and whose values belong to $\mathbb{C}_{\geq}^{q \times q}$ is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$. Let $\mathfrak{B}_{\mathbb{R}}$ (resp. $\mathfrak{B}_{\mathbb{C}}$) be the σ-algebra of all Borel subsets of $\mathbb{R}$ (resp. $\mathbb{C}$). For each $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$, let $\mathfrak{B}_{\Omega} := \mathfrak{B}_{\mathbb{R}} \cap \Omega$, and let $\mathcal{M}_{\geq}^q(\Omega)$ be the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{B}_{\Omega})$. Furthermore, for each $\Omega \in \mathfrak{B}_{\mathbb{R}} \setminus \{\emptyset\}$ and every $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $\mathcal{M}_{\geq,\kappa}^q(\Omega)$ be the set of all $\sigma \in \mathcal{M}_{\geq}^q(\Omega)$ such that the integral $s_j^{(\sigma)} := \int_{\Omega} t^j \sigma(\mathrm{d}t)$ exists for all $j \in \mathbb{Z}_{0,\kappa}$. In this paper, for an arbitrarily fixed $\alpha \in \mathbb{R}$, we study classes of $q \times q$ matrix-valued functions which are holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$. These classes turn out to be closely related via Stieltjes transform with the following truncated matricial Stieltjes type moment problem:

$\mathsf{M}[[\alpha, +\infty); (s_j)_{j=0}^m, =]$ Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, =]$ of all $\sigma \in \mathcal{M}_{\geq,m}^q([\alpha, +\infty))$ for which $s_j^{(\sigma)} = s_j$ is fulfilled for each $j \in \mathbb{Z}_{0,m}$.

In a forthcoming paper, we will indicate how classes of holomorphic functions studied in this paper can be used to parametrize the set $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, =]$.

This paper is organized as follows. In Section 2, we introduce several classes of holomorphic $q \times q$ matrix-valued functions. A particular important role will be played by the class $\mathcal{S}_{q;[\alpha,+\infty)}$ of $[\alpha, +\infty)$-Stieltjes functions of order q, which was considered in the special case $q = 1$ and $\alpha = 0$ by I. S. Kats and M. G. Krein in [17] (see Def. 2.1). In Section 3, we derive several integral representations for functions belonging to $\mathcal{S}_{q;[\alpha,+\infty)}$ (see Theorems 3.1 and 3.6). Furthermore we analyse the structure of ranges and null spaces of the values of functions belonging to $\mathcal{S}_{q;[\alpha,+\infty)}$ (see Prop. 3.15). In Section 4, we state characterizations of the membership of a function to the class $\mathcal{S}_{q;[\alpha,+\infty)}$. In Section 5, we investigate the subclass $\mathcal{S}_{0,q;[\alpha,+\infty)}$ (see Notation 2.7 below) of $\mathcal{S}_{q;[\alpha,+\infty)}$. It is shown in Thm. 5.1 that this class is formed exactly by those $q \times q$ matrix-valued functions defined in $\mathbb{C} \setminus [\alpha, +\infty)$ which can be written as Stieltjes transform of a non-negative Hermitian $q \times q$ measure defined on the Borelian σ-algebra of the interval $[\alpha, +\infty)$. In Section 6, we investigate the Moore-Penrose inverses of the functions belonging to $\mathcal{S}_{q;[\alpha,+\infty)}$. In particular, we show that the Moore-Penrose inverse $F^\dagger$ of a function $F \in \mathcal{S}_{q;[\alpha,+\infty)}$ is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$ and that the function $G \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ defined by $G(z) := -(z - \alpha)^{-1}[F(z)]^\dagger$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ as well (see Thm. 6.3). The second main theme of Section 6 is concerned with the investigation of the class $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ (see Notation 6.5), which was considered in the special case $q = 1$ and $\alpha = 0$ by I. S. Kats

and M. G. Krein in [17]. The main result on the class $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ is Thm. 6.10 which contains an integral representation which is even new for the case $q = 1$ and $\alpha = 0$. The application of Thm. 6.10 enables us to obtain much information about ranges, null spaces and Moore-Penrose inverses of the functions belonging to the class $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ (see Prop. 6.14 and Thm. 6.18). In the remaining sections of this paper, we carry out corresponding investigations for dual classes of $q \times q$ matrix-valued functions which are related to an interval $(-\infty, \beta]$. These classes occur in the treatment of a matrix moment problem $\mathsf{M}[(-\infty, \beta]; (s_j)_{j=0}^m, =]$, which is analogous to Problem $\mathsf{M}[[\alpha, +\infty); (s_j)_{j=0}^m, =]$ formulated above. In Appendix A, we summarize some facts from the integration theory with respect to non-negative Hermitian $q \times q$ measures.

2. On several classes of holomorphic matrix-valued functions

In this section, we introduce those classes of holomorphic $q \times q$ matrix-valued functions which form the central objects of this paper. For each $A \in \mathbb{C}^{q \times q}$, let $\operatorname{Re} A := \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A := \frac{1}{2\mathrm{i}}(A - A^*)$ be the real part of A and the imaginary part of A, respectively. Let $\Pi_+ := \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ and $\Pi_- := \{z \in \mathbb{C} \colon \operatorname{Im} z < 0\}$ be the open upper half plane and open lower half plane of $\mathbb{C}$, respectively. The first two dual classes of holomorphic matrix-valued functions, which are particularly important for this paper, are the following.

Definition 2.1. Let $\alpha \in \mathbb{R}$ and let $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$. Then F is called a $[\alpha, +\infty)$-*Stieltjes function of order* q if F satisfies the following three conditions:

(I) F is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$.

(II) For all $w \in \Pi_+$, the matrix $\operatorname{Im}[F(w)]$ is non-negative Hermitian.

(III) For all $w \in (-\infty, \alpha)$, the matrix $F(w)$ is non-negative Hermitian.

We denote by $\mathcal{S}_{q;[\alpha,+\infty)}$ the set of all $[\alpha, +\infty)$-Stieltjes functions of order q.

Example 2.2. Let $\alpha \in \mathbb{R}$ and let $A, B \in \mathbb{C}_{\succeq}^{q \times q}$. Let $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ be defined by $F(z) := A + \frac{1}{\alpha - z}B$. Since $\operatorname{Im} F(z) = \frac{\operatorname{Im} z}{|\alpha - z|^2}B$ holds true for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, we have $F \in \mathcal{S}_{q;[\alpha,+\infty)}$.

Definition 2.3. Let $\beta \in \mathbb{R}$ and let $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$. Then G is called a $(-\infty, \beta]$-*Stieltjes function of order* q if G fulfills the following three conditions:

(I) G is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$.

(II) For all $w \in \Pi_+$, the matrix $\operatorname{Im}[G(w)]$ is non-negative Hermitian.

(III) For all $w \in (\beta, +\infty)$, the matrix $-G(w)$ is non-negative Hermitian.

We denote by $\mathcal{S}_{q;(-\infty,\beta]}$ the set of all $(-\infty, \beta]$-Stieltjes functions of order q.

Example 2.4. Let $\beta \in \mathbb{R}$ and let $A, B \in \mathbb{C}_{\geq}^{q \times q}$. Let $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ be defined by $G(z) := -A + \frac{1}{\beta - z} B$. Since $\operatorname{Im} G(z) = \frac{\operatorname{Im} z}{|\beta - z|^2} B$ holds true for each $z \in \mathbb{C} \setminus (-\infty, \beta]$, we have $G \in \mathcal{S}_{q;(-\infty,\beta]}$.

The particular functions belonging to the class $\mathcal{S}_{q;[\alpha,+\infty)}$ resp. $\mathcal{S}_{q;(-\infty,\beta]}$, which were introduced in Example 2.2 (resp. Example 2.4), were called by V. E. Katsnelson [18] *special* functions belonging to $\mathcal{S}_{q;[\alpha,+\infty)}$ (resp. $\mathcal{S}_{q;(-\infty,\beta]}$), whereas all remaining functions contained in $\mathcal{S}_{q;[\alpha,+\infty)}$ (resp. $\mathcal{S}_{q;(-\infty,\beta]}$) were called *generic* functions belonging to $\mathcal{S}_{q;[\alpha,+\infty)}$ (resp. $\mathcal{S}_{q;(-\infty,\beta]}$). In the case $q = 1$ and $\alpha = 0$ the general theory of multiplicative representations of functions belonging to $\mathcal{S}_{1;[0,+\infty)}$ was treated in detail by Aronszajn/Donoghue [2].

It should be mentioned that Yu. M. Dyukarev and V. E. Katsnelson studied in [6–8] an interpolation problem for functions belonging to the class $\mathcal{S}_{q;(-\infty,0]}$. Their approach was based on V. P. Potapov's method of fundamental matrix inequalities. V. E. Katsnelson [18] used the class $\mathcal{S}_{1;(-\infty,0]}$ to construct Hurwitz stable entire functions.

First more or less obvious interrelations between the above introduced classes of functions can be described by the following result. For each $\alpha \in \mathbb{R}$, let the mapping $T_\alpha \colon \mathbb{R} \to \mathbb{R}$ be defined by $T_\alpha(x) := x + \alpha$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are non-empty sets with $\mathcal{Z} \subseteq \mathcal{X}$ and if $f \colon \mathcal{X} \to \mathcal{Y}$ is a mapping, then we will use $\operatorname{Rstr}_{\mathcal{Z}} f$ to denote the restriction of f onto $\mathcal{Z}$.

Remark 2.5. (a) If $\alpha \in \mathbb{R}$ and if $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$, then F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ if and only if the function $F \circ \operatorname{Rstr}_{\mathbb{C} \setminus [0,+\infty)} T_\alpha$ belongs to the class $\mathcal{S}_{q;[0,+\infty)}$.

 (b) If $\alpha \in \mathbb{R}$ and if $F \colon \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}^{q \times q}$, then $F \in \mathcal{S}_{q;[0,+\infty)}$ if and only if $F \circ \operatorname{Rstr}_{\mathbb{C} \setminus [\alpha,+\infty)} T_{-\alpha} \in \mathcal{S}_{q;[\alpha,+\infty)}$.

Remark 2.6. (a) If $\beta \in \mathbb{R}$ and if $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$, then G belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ if and only if the function $G \circ \operatorname{Rstr}_{\mathbb{C} \setminus (-\infty,0]} T_\beta$ belongs to the class $\mathcal{S}_{q;(-\infty,0]}$.

 (b) If $\beta \in \mathbb{R}$ and if $G \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}^{q \times q}$, then $F \in \mathcal{S}_{q;(-\infty,0]}$ if and only if $F \circ \operatorname{Rstr}_{\mathbb{C} \setminus (-\infty,\beta]} T_{-\beta} \in \mathcal{S}_{q;(-\infty,\beta]}$.

Now we introduce particular subclasses of $\mathcal{S}_{q;[\alpha,+\infty)}$ and $\mathcal{S}_{q;(-\infty,\beta]}$, which will turn out to be important in studying matricial versions of the Stieltjes moment problem. If $A \in \mathbb{C}^{p \times q}$, then we denote by $\|A\|_{\mathrm{E}} := \sqrt{\operatorname{tr}(A^*A)}$ the Euclidean norm of A. Special attention will be put in the sequel also to the following subclasses of the classes of holomorphic matrix-valued functions introduced in Definitions 2.1 and 2.3.

Notation 2.7. Let $\alpha \in \mathbb{R}$. Then let $\mathcal{S}_{0,q;[\alpha,+\infty)}$ be the class of all S which belong to $\mathcal{S}_{q;[\alpha,+\infty)}$ and which satisfy

$$\sup_{y \in [1,+\infty)} y \|S(\mathrm{i}y)\|_{\mathrm{E}} < +\infty. \tag{2.1}$$

Furthermore, let $\mathcal{S}_{0,q;(-\infty,\alpha]}$ be the class of all $S \in \mathcal{S}_{q;(-\infty,\alpha]}$ which satisfy (2.1).

Remark 2.8. Let $\alpha \in \mathbb{R}$. If $S \in \mathcal{S}_{0,q;[\alpha,+\infty)}$ or if $S \in \mathcal{S}_{0,q;(-\infty,\alpha]}$, then $\lim_{y \to +\infty} S(\mathrm{i}y) = O_{q \times q}$.

Rem. 2.8 leads us to the following classes, which will play an important role in the framework of a truncated matricial Stieltjes moment problems.

Notation 2.9. Let $\alpha \in \mathbb{R}$. Then by $\mathcal{S}^{\diamond}_{q;[\alpha,+\infty)}$ (resp. $\mathcal{S}^{\diamond}_{q;(-\infty,\alpha]}$) we denote the set of all $F \in \mathcal{S}_{q;[\alpha,+\infty)}$ (resp. $\mathcal{S}^{\diamond}_{q;(-\infty,\alpha]}$) which satisfy $\lim_{y\to+\infty} F(iy) = O_{q\times q}$.

Remark 2.10. If $\alpha \in \mathbb{R}$, then Rem. 2.8 shows that $\mathcal{S}_{0,q;[\alpha,+\infty)} \subseteq \mathcal{S}^{\diamond}_{q;[\alpha,+\infty)}$ and $\mathcal{S}_{0,q;(-\infty,\alpha]} \subseteq \mathcal{S}^{\diamond}_{q;(-\infty,\alpha]}$.

Remark 2.11. Let $\alpha \in \mathbb{R}$ and let

$$\mathcal{S} \in \{\mathcal{S}_{q;[\alpha,+\infty)}, \mathcal{S}_{0,q;[\alpha,+\infty)}, \mathcal{S}^{\diamond}_{q;[\alpha,+\infty)}, \mathcal{S}_{q;(-\infty,\alpha]}, \mathcal{S}_{0,q;(-\infty,\alpha]}, \mathcal{S}^{\diamond}_{q;(-\infty,\alpha]}\}.$$

Then $F \in \mathcal{S}$ if and only if $F^{\mathrm{T}} \in \mathcal{S}$.

Remark 2.12. (a) Let $\alpha \in \mathbb{R}$ and let $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Then it is readily checked that F belongs to one of the classes $\mathcal{S}_{q;[\alpha,+\infty)}$, $\mathcal{S}_{0,q;[\alpha,+\infty)}$, and $\mathcal{S}^{\diamond}_{q;[\alpha,+\infty)}$ if and only if for each $u \in \mathbb{C}^q$ the function $u^* F u$ belongs to the corresponding classes $\mathcal{S}_{1;[\alpha,+\infty)}$, $\mathcal{S}_{0,1;[\alpha,+\infty)}$, and $\mathcal{S}^{\diamond}_{1;[\alpha,+\infty)}$, respectively.

(b) Let $\beta \in \mathbb{R}$ and let $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Then it is readily checked that G belongs to one of the classes $\mathcal{S}_{q;(-\infty,\beta]}$, $\mathcal{S}_{0,q;(-\infty,\beta]}$, and $\mathcal{S}^{\diamond}_{q;(-\infty,\beta]}$ if and only if for each $u \in \mathbb{C}^q$ the function $u^* G u$ belongs to the corresponding classes $\mathcal{S}_{1;(-\infty,\beta]}$, $\mathcal{S}_{0,1;(-\infty,\beta]}$, and $\mathcal{S}^{\diamond}_{1;(-\infty,\beta]}$, respectively.

In order to get integral representations and other useful information about the one-sided Stieltjes functions of order q, we exploit the fact that these classes of functions can be embedded via restriction to the upper half plane Π_+ to the well-studied Herglotz-Nevanlinna class $\mathcal{R}_q(\Pi_+)$ of all Herglotz-Nevanlinna functions in Π_+. A matrix-valued function $F \colon \Pi_+ \to \mathbb{C}^{q\times q}$ is called a $q \times q$ *Herglotz-Nevanlinna function in* Π_+ if F is holomorphic in Π_+ and satisfies the condition $\mathrm{Im}[F(w)] \in \mathbb{C}^{q\times q}_{\geq}$ for all $w \in \Pi_+$. For a comprehensive study on the class $\mathcal{R}_q(\Pi_+)$, we refer the reader to the paper [16] by F. Gesztesy and E. R. Tsekanovskii and to the paper [11]. In particular, in [11] one can find a detailed discussion of the holomorphicity properties of the Moore-Penrose pseudoinverse of matrix-valued Herglotz-Nevanlinna functions. Before we recall the well-known characterization of the class $\mathcal{R}_q(\Pi_+)$ via Nevanlinna parametrization, we observe that, for each $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ and each $z \in \mathbb{C} \setminus \mathbb{R}$, the function $h_z \colon \mathbb{R} \to \mathbb{C}$ defined by $h_z(t) := (1 + tz)/(t - z)$ belongs to $\mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \nu; \mathbb{C})$.

Theorem 2.13. (a) Let $F \in \mathcal{R}_q(\Pi_+)$. Then there are unique matrices $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$ and $B \in \mathbb{C}^{q\times q}_{\geq}$ and a unique non-negative Hermitian measure $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ such that

$$F(z) = A + zB + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \nu(\mathrm{d}t) \qquad \text{for each } z \in \Pi_+. \qquad (2.2)$$

(b) If $A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$, if $B \in \mathbb{C}^{q\times q}_{\geq}$, and if $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R})$, then $F \colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by (2.2) belongs to the class $\mathcal{R}_q(\Pi_+)$.

For each $F \in \mathcal{R}_q(\Pi_+)$, the unique triple $(A, B, \nu) \in \mathbb{C}_{\mathrm{H}}^{q \times q} \times \mathbb{C}_{\geq}^{q \times q} \times \mathcal{M}_{\geq}^q(\mathbb{R})$ for which the representation (2.2) holds true is called the *Nevanlinna parametrization of F* and we will also write (A_F, B_F, ν_F) for (A, B, ν).

Remark 2.14. Let $F \in \mathcal{R}_q(\Pi_+)$ with Nevanlinna parametrization (A_F, B_F, ν_F). Then it is immediately seen that F^{T} belongs to $\mathcal{R}_q(\Pi_+)$ and that $(A_{F^{\mathrm{T}}}, B_{F^{\mathrm{T}}}, \nu_{F^{\mathrm{T}}}) = (A_F^{\mathrm{T}}, B_F^{\mathrm{T}}, \nu_F^{\mathrm{T}})$.

Remark 2.15. Let $F \in \mathcal{R}_q(\Pi_+)$ with Nevanlinna parametrization (A, B, ν). In view of Thm. 2.13, we have then $-\overline{z} \in \Pi_+$ and

$$-[F(-\overline{z})]^* = -A + zB + \int_{\mathbb{R}} \frac{1 - tz}{-t - z} \nu(\mathrm{d}t) = -A + zB + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \theta(\mathrm{d}t)$$

for all $z \in \Pi_+$, where θ is the image measure of ν under the reflection $t \mapsto -t$ on $\mathbb{R}$. Because of $-A \in \mathbb{C}_{\mathrm{H}}^{q \times q}$ and $B \in \mathbb{C}_{\geq}^{q \times q}$, Thm. 2.13 yields then that $G \colon \Pi_+ \to \mathbb{C}^{q \times q}$ defined by $G(z) := -[F(-\overline{z})]^*$ belongs to $\mathcal{R}_q(\Pi_+)$ and that the Nevanlinna parametrization of G is given by $(-A, B, \theta)$.

From Definitions 2.1 and 2.3 we see immediately that $\{\mathrm{Rstr}_{\Pi_+} F \colon F \in \mathcal{S}_{q;[\alpha, +\infty)}\} \subseteq \mathcal{R}_q(\Pi_+)$ and $\{\mathrm{Rstr}_{\Pi_+} G \colon G \in \mathcal{S}_{q;(-\infty, \beta]}\} \subseteq \mathcal{R}_q(\Pi_+)$. Now we analyse the Nevanlinna parametrizations of the restrictions on Π_+ of the members of these classes of functions.

Proposition 2.16. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha, +\infty)}$. Then the Nevanlinna parametrization (A, B, ν) of $\mathrm{Rstr}_{\Pi_+} F$ fulfills*

$$\nu((-\infty, \alpha)) = O_{q \times q}, \qquad B = O_{q \times q}, \qquad \text{and} \qquad \nu \in \mathcal{M}_{\geq, 1}^q(\mathbb{R}). \qquad (2.3)$$

In particular, for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then

$$F(z) = A + \int_{[\alpha, +\infty)} \frac{1 + tz}{t - z} \nu(\mathrm{d}t). \qquad (2.4)$$

Proof. From [4, Prop. 8.3] and its proof we obtain $\nu((-\infty, \alpha)) = O_{q \times q}$ and

$$F(z) = A + zB + \int_{[\alpha, +\infty)} \frac{1 + tz}{t - z} \nu(\mathrm{d}t) \qquad (2.5)$$

for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Let $u \in \mathbb{C}^q$. Then $u^* \nu u \in \mathcal{M}_{\geq}^1(\mathbb{R})$ and (2.5) yields

$$u^* F(x) u = u^* A u + x u^* B u + \int_{[\alpha, +\infty)} \frac{1 + tx}{t - x} (u^* \nu u)(\mathrm{d}t) \qquad (2.6)$$

for all $x \in (-\infty, \alpha)$. Let $\alpha_1 := \min\{\alpha - 1, -1\}$ and $\alpha_2 := \max\{\alpha + 1, 1\}$. Since $F(x) \in \mathbb{C}_{\geq}^{q \times q}$ for all $x \in (-\infty, \alpha_1)$, and furthermore $(1 + tx)/(t - x) < 0$ for all $x \in (-\infty, \alpha_1)$ and all $t \in (\alpha_2, +\infty)$, we conclude from (2.6) then

$$-x u^* B u \leq u^* A u + \int_{[\alpha, \alpha_2]} \frac{1 + tx}{t - x} (u^* \nu u)(\mathrm{d}t)$$

for all $x \in (-\infty, \alpha_1)$. One can easily check that there exists a constant $L_\alpha \in \mathbb{R}$ depending only on α such that $|(1 + tx)/(t - x)| \leq L_\alpha$ for all $x \in (-\infty, \alpha_1)$ and all $t \in [\alpha, \alpha_2]$. For all $x \in (-\infty, \alpha_1)$, hence

$$\left| \int_{[\alpha,\alpha_2]} \frac{1 + tx}{t - x} (u^* \nu u)(\mathrm{d}t) \right| \leq L_\alpha \cdot (u^* \nu u)([\alpha, \alpha_2]) < +\infty. \tag{2.7}$$

Setting $K := u^* A u + L_\alpha \cdot (u^* \nu u)([\alpha, \alpha_2])$, we get then $-x u^* B u \leq K < +\infty$. In view of $B \in \mathbb{C}_\geq^{q \times q}$, we have $u^* B u \geq 0$, where $u^* B u > 0$ is impossible, since $-x u^* B u$ would then tend to $+\infty$ as x tends to $-\infty$. Thus, $u^* B u = 0$. Since $u \in \mathbb{C}^q$ was arbitrarily chosen, we get $B = O_{q \times q}$ and, in view of (2.5), thus (2.4) holds true for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Taking additionally into account $F(x) \in \mathbb{C}_\geq^{q \times q}$ for all $x \in (-\infty, \alpha_1)$, we conclude from (2.6) and (2.7) furthermore $\int_{[\alpha_2,+\infty)} [-(1 + tx)/(t - x)](u^* \nu u)(\mathrm{d}t) \leq K < +\infty$ for all $x \in (-\infty, \alpha_1)$. Now we consider an arbitrary sequence $(x_n)_{n=1}^\infty$ from $(-\infty, \alpha_1)$ with $\lim_{n \to \infty} x_n = -\infty$. Then $-(1 + tx_n)/(t - x_n) > 0$ for all $n \in \mathbb{N}$ and all $t \in [\alpha_2, +\infty)$ and, furthermore, $|t| = \liminf_{n \to \infty} [-(1 + tx_n)/(t - x_n)]$ for all $t \in [\alpha_2, +\infty)$. The application of Fatou's lemma then yields

$$\int_{[\alpha_2,+\infty)} |t| (u^* \nu u)(\mathrm{d}t) \leq \liminf_{n \to \infty} \int_{[\alpha_2,+\infty)} \left(-\frac{1 + tx}{t - x} \right) (u^* \nu u)(\mathrm{d}t) \leq K < +\infty.$$

Since $(u^* \nu u)((-\infty, \alpha)) = 0$ and since $\int_{[\alpha,\alpha_2]} |t| (u^* \nu u)(\mathrm{d}t)$ is finite, we conclude then $\int_{\mathbb{R}} |t| (u^* \nu u)(\mathrm{d}t) < +\infty$. Because $u \in \mathbb{C}^q$ was arbitrarily chosen, we get $\nu \in \mathcal{M}_{\geq,1}^q(\mathbb{R})$. $\quad\square$

3. Integral representations for the class $\mathcal{S}_{q;[\alpha,+\infty)}$

The main goal of this section is to derive some integral representations for $[\alpha, +\infty)$-Stieltjes functions of order q.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ and let $F : \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$. Then:*

(a) *Suppose $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. Denote by (A, B, ν) the Nevanlinna parametrization of $\mathrm{Rstr}_{\Pi_+} F$. Then $\tilde{\nu} := \mathrm{Rstr}_{\mathfrak{B}_{[\alpha,+\infty)}} \nu$ belongs to $\mathcal{M}_{\geq,1}^q([\alpha, +\infty))$ and there is a unique pair (C, η) belonging to $\mathbb{C}_\geq^{q \times q} \times \mathcal{M}_{\geq,1}^q([\alpha, +\infty))$ such that*

$$F(z) = C + \int_{[\alpha,+\infty)} \frac{1 + t^2}{t - z} \eta(\mathrm{d}t) \tag{3.1}$$

for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, namely $C = A - \int_{[\alpha,+\infty)} t \tilde{\nu}(\mathrm{d}t)$ and $\eta = \tilde{\nu}$. Furthermore, $C = \lim_{r \to +\infty} F(\alpha + re^{\mathrm{i}\phi})$ for all $\phi \in (\pi/2, 3\pi/2)$.

(b) *Let $C \in \mathbb{C}_\geq^{q \times q}$ and let $\eta \in \mathcal{M}_{\geq,1}^q([\alpha, +\infty))$ be such that (3.1) holds true for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Then F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$.*

Proof. (a) From Prop. 2.16 we conclude $\tilde{\nu} \in \mathcal{M}^q_{\geq,1}([\alpha,+\infty))$. Let $u \in \mathbb{C}^q$. Then $\int_{[\alpha,+\infty)} t(u^*\tilde{\nu}u)(\mathrm{d}t)$ is finite. Obviously, for all $z \in \mathbb{C}$ and all $t \in \mathbb{R}$ with $t \neq z$, we get

$$\frac{1+tz}{t-z} + t = \frac{1+t^2}{t-z}. \tag{3.2}$$

Thus, in view of Prop. 2.16, we obtain

$$\int_{[\alpha,+\infty)} \frac{1+t^2}{t-z}(u^*\tilde{\nu}u)(\mathrm{d}t) < +\infty \tag{3.3}$$

and

$$u^*F(z)u = u^*Au + \int_{[\alpha,+\infty)} \frac{1+t^2}{t-z}(u^*\tilde{\nu}u)(\mathrm{d}t) - \int_{[\alpha,+\infty)} t(u^*\tilde{\nu}u)(\mathrm{d}t) \tag{3.4}$$

for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Since $u \in \mathbb{C}^q$ was arbitrarily chosen, it follows (3.1) for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$ with $C := A - \int_{[\alpha,+\infty)} t\tilde{\nu}(\mathrm{d}t)$ and $\eta := \tilde{\nu}$. Let $\phi \in (\pi/2, 3\pi/2)$. Then $\cos\phi < 0$. To show $C = \lim_{r\to+\infty} F(\alpha + re^{i\phi})$, we consider an arbitrary sequence $(r_n)_{n=1}^{\infty}$ from $\mathbb{R}$ with $r_n \geq 1/|\cos\phi|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = +\infty$. We have then $\lim_{n\to\infty}(1+t^2)/(t-\alpha-r_ne^{i\phi}) = 0$ for all $t \in [\alpha,+\infty)$. For all $n \in \mathbb{N}$ and all $t \in [\alpha,+\infty)$, we get furthermore

$$|t-\alpha-r_ne^{i\phi}| \geq t-\alpha-r_n\cos\phi = t-\alpha+r_n|\cos\phi| \geq t-\alpha+1 \geq 1$$

and hence $|(1+t^2)/(t-\alpha-r_ne^{i\phi})| \leq (1+t^2)/(t-\alpha+1)$. Since, because of (3.3), the integral $\int_{[\alpha,+\infty)}(1+t^2)/(t-\alpha+1)(u^*\tilde{\nu}u)(\mathrm{d}t)$ is finite, the application of Lebesgue's dominated convergence theorem yields $\lim_{n\to\infty}\int_{[\alpha,+\infty)}(1+t^2)/(t-\alpha-r_ne^{i\phi})(u^*\tilde{\nu}u)(\mathrm{d}t) = 0$. From (3.4) we conclude then $u^*Cu = \lim_{n\to\infty} u^*F(\alpha+r_ne^{i\phi})u$. Since $u \in \mathbb{C}^q$ was arbitrarily chosen, we obtain $C = \lim_{n\to\infty} F(\alpha+r_ne^{i\phi})$. Taking into account $F(x) \in \mathbb{C}^{q\times q}_{\geq}$ for all $x \in (-\infty,\alpha)$, we get with $\phi = \pi$ in particular $C \in \mathbb{C}^{q\times q}_{\geq}$.

Now let $C \in \mathbb{C}^{q\times q}_{\geq}$ and $\eta \in \mathcal{M}^q_{\geq,1}([\alpha,+\infty))$ be such that (3.1) holds true for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Then $\chi\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}_{\geq}$ defined by $\chi(M) := \eta(M \cap [\alpha,+\infty))$ belongs to $\mathcal{M}^q_{\geq,1}(\mathbb{R})$ and the matrix $C + \int_{\mathbb{R}} t\chi(\mathrm{d}t)$ is Hermitian. Using (3.2), we infer from (3.1) that the integral $\int_{\mathbb{R}}(1+tz)/(t-z)\chi(\mathrm{d}t)$ exists and that

$$F(z) = C + \int_{\mathbb{R}} t\chi(\mathrm{d}t) + z \cdot O_{q\times q} + \int_{\mathbb{R}} \frac{1+tz}{t-z}\chi(\mathrm{d}t)$$

is fulfilled for all $z \in \Pi_+$. Thm. 2.13(a) yields then $C + \int_{\mathbb{R}} t\chi(\mathrm{d}t) = A$ and $\chi = \nu$. Hence $\eta = \tilde{\nu}$ and $C = A - \int_{[\alpha,+\infty)} t\tilde{\nu}(\mathrm{d}t)$.

(b) Let $C \in \mathbb{C}^{q\times q}_{\geq}$ and $\eta \in \mathcal{M}^q_{\geq,1}([\alpha,+\infty))$ be such that (3.1) holds true for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Using a result on holomorphic dependence of an integral on a complex parameter (see, e.g. [9, Ch. IV, §5, Satz 5.8]), we conclude then that F is a matrix-valued function which is holomorphic in $\mathbb{C} \setminus [\alpha,+\infty)$. Furthermore,

$$\operatorname{Im} F(z) = \int_{[\alpha,+\infty)} \operatorname{Im}\left(\frac{1+t^2}{t-z}\right)\eta(\mathrm{d}t) = \int_{[\alpha,+\infty)} \frac{(1+t^2)\operatorname{Im} z}{|t-z|^2}\eta(\mathrm{d}t) \in \mathbb{C}^{q\times q}_{\geq}$$

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for all $z \in \Pi_+$ and

$$F(x) = C + \int_{[\alpha,+\infty)} \frac{1+t^2}{t-x} \eta(\mathrm{d}t) \in \mathbb{C}_{\geq}^{q\times q}$$

for all $x \in (-\infty, \alpha)$. Thus, F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$. $\qquad \square$

In the following, if $\alpha \in \mathbb{R}$ and $F \in \mathcal{S}_{q;[\alpha,+\infty)}$ are given, then we will write (C_F, η_F) for the unique pair (C, η) belonging to $\mathbb{C}_{\geq}^{q\times q} \times \mathcal{M}_{\geq,1}^q([\alpha,+\infty))$ which fulfills (3.1) for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Furthermore, if A and B are complex $q \times q$ matrices, then we write $A \leq B$ or $B \geq A$ to indicate that the matrices A and B are Hermitian and that $B - A$ is non-negative Hermitian.

Remark 3.2. Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. For all $x_1, x_2 \in (-\infty, \alpha)$ with $x_1 \leq x_2$, then $O_{q\times q} \leq F(x_1) \leq F(x_2)$, by virtue of Thm. 3.1(a).

Remark 3.3. Let $\alpha \in \mathbb{R}$ and let $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Then, for each $\mu \in \mathcal{M}_{\geq}^q([\alpha,+\infty))$, in view of the equation $(1 + t - \alpha)/(t - z) = 1 + (1 + z - \alpha)/(t - z)$, which holds for each $t \in [\alpha,+\infty)$, and Lem. A.8(a), one can easily see that the function $h_{\alpha,z} \colon [\alpha,+\infty) \to \mathbb{C}$ defined by $h_{\alpha,z}(t) := (1 + t - \alpha)/(t - z)$ belongs to $\mathcal{L}^1([\alpha,+\infty), \mathfrak{B}_{[\alpha,+\infty)}, \mu; \mathbb{C})$.

Lemma 3.4. *Let* $\alpha \in \mathbb{R}$ *and let* $F \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q\times q}$ *be a continuous matrix-valued function such that* $(\operatorname{Im} F)(\Pi_+) \subseteq \mathbb{C}_{\geq}^{q\times q}$ *and* $(-\operatorname{Im} F)(\Pi_-) \subseteq \mathbb{C}_{\geq}^{q\times q}$. *Then* $F(x) = \operatorname{Re} F(x)$ *and* $\operatorname{Im} F(x) = O_{q\times q}$ *for each* $x \in (-\infty, \alpha)$.

Proof. Let $x \in (-\infty, \alpha)$. Then $(\operatorname{Im} F(x + \mathrm{i}/n))_{n=1}^{\infty}$ and $(-\operatorname{Im} F(x - \mathrm{i}/n))_{n=1}^{\infty}$ are sequences of non-negative Hermitian complex $q \times q$ matrices which converge to the non-negative Hermitian complex matrices $\operatorname{Im} F(x)$ and $-\operatorname{Im} F(x)$, respectively. Consequently, $\operatorname{Im} F(x) = O_{q\times q}$, which implies $F(x) = \operatorname{Re} F(x)$. $\qquad \square$

Lemma 3.5. *Let* $\alpha \in \mathbb{R}$, *let* $\gamma \in \mathbb{C}_{\geq}^{q\times q}$, *and let* $\mu \in \mathcal{M}_{\geq}^q([\alpha,+\infty))$. *Let* $F \colon \mathbb{C}\setminus[\alpha,+\infty) \to \mathbb{C}^{q\times q}$ *be defined by*

$$F(z) = \gamma + \int_{[\alpha,+\infty)} \frac{1+t-\alpha}{t-z} \mu(\mathrm{d}t). \tag{3.5}$$

Then

$$\operatorname{Re} F(z) = \gamma + \int_{[\alpha,+\infty)} \frac{1+t-\alpha}{|t-z|^2}(t - \operatorname{Re} z)\mu(\mathrm{d}t) \tag{3.6}$$

and

$$\operatorname{Im} F(z) = (\operatorname{Im} z) \int_{[\alpha,+\infty)} \frac{1+t-\alpha}{|t-z|^2} \mu(\mathrm{d}t) \tag{3.7}$$

hold true for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$. *In particular,*

$$(\operatorname{Re} F)(\mathbb{C}_{\alpha,-}) \subseteq \mathbb{C}_{\geq}^{q\times q}, \qquad (\operatorname{Im} F)(\Pi_+) \subseteq \mathbb{C}_{\geq}^{q\times q}, \qquad and \qquad (-\operatorname{Im} F)(\Pi_-) \subseteq \mathbb{C}_{\geq}^{q\times q}.$$

Furthermore, $F(x) = \operatorname{Re} F(x)$ *and* $F(x) \in \mathbb{C}_{\geq}^{q\times q}$ *for each* $x \in (-\infty, \alpha)$.

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Proof. For each $z \in \mathbb{C} \setminus [\alpha, +\infty)$ and each $t \in [\alpha, +\infty)$, we have

$$\operatorname{Re}\left(\frac{1+t-\alpha}{t-z}\right) = \frac{1+t-\alpha}{|t-z|^2}(t - \operatorname{Re} z) \tag{3.8}$$

and $\operatorname{Im}[(1+t-\alpha)/(t-z)] = (\operatorname{Im} z)(1+t-\alpha)/|t-z|^2$. Taking into account $\gamma \in \mathbb{C}_{\geq}^{q \times q}$, thus (3.6) and (3.7) follow for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$. For each $z \in \mathbb{C}_{\alpha,-}$ and each $t \in [\alpha, +\infty)$, the right-hand side of (3.8) belongs to $[0, +\infty)$. Thus, $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and (3.6) show that $\operatorname{Re} F(z)$ belongs to $\mathbb{C}_{\geq}^{q \times q}$ for each $z \in \mathbb{C}_{\alpha,-}$. Since $(1+t-\alpha)/|t-z|^2 \in [0, +\infty)$ for every choice of $z \in \mathbb{C} \setminus [\alpha, +\infty)$ and $t \in [\alpha, +\infty)$, from (3.7) we see that $\operatorname{Im} F(w) \in \mathbb{C}_{\geq}^{q \times q}$ for each $w \in \Pi_+$ and $-\operatorname{Im} F(v) \in \mathbb{C}_{\geq}^{q \times q}$ for each $v \in \Pi_-$ are fulfilled. Applying Lem. 3.4 completes the proof. $\qquad\square$

Now we give a further integral representation of the matrix-valued functions which belong to the class $\mathcal{S}_{q;[\alpha,+\infty)}$. In the special case that $q = 1$ and $\alpha = 0$ hold, one can find this result in [19, Appendix].

Theorem 3.6. *Let $\alpha \in \mathbb{R}$ and let $F: \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$. Then:*

(a) *If $F \in \mathcal{S}_{q;[\alpha,+\infty)}$, then there are a unique matrix $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and a unique non-negative Hermitian measure $\mu \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ such that (3.5) holds true for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Furthermore, $\gamma = \overline{C}_F$.*

(b) *If there are a matrix $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and a non-negative Hermitian measure $\mu \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ such that F can be represented via (3.5) for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then F belongs to the class $\mathcal{S}_{q;[\alpha,+\infty)}$.*

Proof. Denote $f := F \circ \operatorname{Rstr}_{\mathbb{C} \setminus [0,+\infty)} T_\alpha$.

(a) Let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. According to Rem. 2.5, the function f belongs to $\mathcal{S}_{q;[0,+\infty)}$. In view of Prop. 2.16, then $\operatorname{Rstr}_{\Pi_+} f$ belongs to $\mathcal{R}_q(\Pi_+)$, the Nevanlinna parametrization (A, B, ν) of $\operatorname{Rstr}_{\Pi_+} f$ fulfills (2.3), and, for each $w \in \mathbb{C} \setminus [0, +\infty)$, we have $f(w) = A + \int_{[0,+\infty)} (1+xw)/(x-w)\nu(\mathrm{d}x)$. Because of (2.3), the integral $\int_{[0,+\infty)} x\nu(\mathrm{d}x)$ exists and the mapping $\hat{\mu}: \mathfrak{B}_{[0,+\infty)} \to \mathbb{C}_{\geq}^{q \times q}$ given by $\hat{\mu}(B) := \int_B (1+x^2)/(1+x)\nu(\mathrm{d}x)$ is well defined and belongs to $\mathcal{M}_{\geq}^q([0, +\infty))$. Setting $\gamma := A - \int_{[0,+\infty)} x\nu(\mathrm{d}x)$, for each $w \in \mathbb{C} \setminus [0, +\infty)$, we get

$$\begin{aligned}
f(w) &= A + \int_{[0,+\infty)} \left(\frac{1+x^2}{x-w} - x\right)\nu(\mathrm{d}x) \\
&= A - \int_{[0,+\infty)} x\nu(\mathrm{d}x) + \int_{[0,+\infty)} \left(\frac{1+x}{x-w} \cdot \frac{1+x^2}{1+x}\right)\nu(\mathrm{d}x) \\
&= \gamma + \int_{[0,+\infty)} \frac{1+x}{x-w}\hat{\mu}(\mathrm{d}x).
\end{aligned}$$

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Obviously, $\mu := (\text{Rstr}_{[0,+\infty)} T_\alpha)(\hat{\mu})$ belongs to $\mathcal{M}^q_{\geq}([\alpha,+\infty))$. For each $z \in \mathbb{C} \setminus [\alpha,+\infty)$, Prop. A.5 yields

$$F(z) = f(z - \alpha) = \gamma + \int_{[0,+\infty)} \frac{1 + x + \alpha - \alpha}{x + \alpha - z} \hat{\mu}(\mathrm{d}x)$$

$$= \gamma + \int_{[0,+\infty)} (h_{\alpha,z} \circ T_\alpha)(x)\hat{\mu}(\mathrm{d}x) = \gamma + \int_{T_\alpha([0,+\infty))} h_{\alpha,z}\mathrm{d}\mu$$

and, hence, (3.5) holds for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$.

Now we assume that γ is an arbitrary complex $q \times q$ matrix and that μ is an arbitrary non-negative Hermitian measure belonging to $\mathcal{M}^q_{\geq}([\alpha,+\infty))$ such that (3.5) holds for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Observe that $\lim_{n \to +\infty} h_{\alpha,\alpha-1-n}(t) = 0$ is valid for each $t \in [\alpha,+\infty)$. Moreover, for every choice of $n \in \mathbb{N}_0$ and $t \in [\alpha,+\infty)$, one can easily check that the estimation $|h_{\alpha,\alpha-1-n}(t)| \leq 1$ holds. Consequently, a matrix generalization of Lebesgue's dominated convergence theorem (see Prop. A.6) yields

$$\lim_{n \to +\infty} \int_{[\alpha,+\infty)} \frac{1 + t - \alpha}{t - (\alpha - 1 - n)} \mu(\mathrm{d}t) = 0.$$

From (3.5) we obtain then

$$\gamma = \gamma + \lim_{n \to +\infty} \int_{[\alpha,+\infty)} \frac{1 + t - \alpha}{t - (\alpha - 1 - n)} \mu(\mathrm{d}t) = \lim_{n \to +\infty} F(\alpha - 1 - n).$$

From Thm. 3.1(a) we conclude then $\gamma = C_F$ and thus $\gamma \in \mathbb{C}^{q \times q}_{\geq}$ follows. The mapping $\tilde{T}_{-\alpha}: [\alpha,+\infty) \to [0,+\infty)$ defined by $\tilde{T}_{-\alpha}(t) := t - \alpha$ is obviously bijective and $\mathfrak{B}_{[\alpha,+\infty)}$-$\mathfrak{B}_{[0,+\infty)}$-measurable. Further, the mapping $\rho: \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q \times q}_{\geq}$ given by $\rho(B) := \int_{B \cap [0,+\infty)} (1 + x)/(1 + x^2)[\tilde{T}_{-\alpha}(\mu)](\mathrm{d}x)$ is well defined, belongs to $\mathcal{M}^q_{\geq}(\mathbb{R})$, and satisfies $\rho((-\infty, 0)) = O_{q \times q}$. Furthermore, the integral $\int_{[0,+\infty)} x\rho(\mathrm{d}x)$ exists. For each $w \in \mathbb{C} \setminus [0,+\infty)$, using (3.5), the relation $\tilde{T}_{-\alpha}([\alpha,+\infty)) = [0,+\infty)$ and Prop. A.5 provide us

$$f(w) = F(w + \alpha) = \gamma + \int_{[\alpha,+\infty)} \frac{1 + t - \alpha}{t - (w + \alpha)} \mu(\mathrm{d}t)$$

$$= \gamma + \int_{[\alpha,+\infty)} \frac{1 + \tilde{T}_{-\alpha}(t)}{\tilde{T}_{-\alpha}(t) - w} \mu(\mathrm{d}t) = \gamma + \int_{[0,+\infty)} \frac{1 + x}{x - w} \left[\tilde{T}_{-\alpha}(\mu)\right](\mathrm{d}x)$$

$$= \gamma + \int_{[0,+\infty)} \left(\frac{1 + x^2}{x - w} \cdot \frac{1 + x}{1 + x^2}\right) \left[\tilde{T}_{-\alpha}(\mu)\right](\mathrm{d}x) \tag{3.9}$$

$$= \gamma + \int_{[0,+\infty)} \frac{1 + x^2}{x - w} \rho(\mathrm{d}x) = \gamma + \int_{[0,+\infty)} \left(x + \frac{1 + xw}{x - w}\right) \rho(\mathrm{d}x)$$

$$= \gamma + \int_{[0,+\infty)} x\rho(\mathrm{d}x) + \int_{[0,+\infty)} \frac{1 + xw}{x - w} \rho(\mathrm{d}x)$$

and, consequently,

$$\text{Rstr}_{\Pi_+} f(w) = \gamma + \int_{[0,+\infty)} x\rho(\mathrm{d}x) + w \cdot O_{q \times q} + \int_{\mathbb{R}} \frac{1 + xw}{x - w} \rho(\mathrm{d}x)$$

for each $w \in \Pi_+$. Since γ is non-negative Hermitian we see that $A^\diamond := \gamma + \int_{[0,+\infty)} x\rho(\mathrm{d}x)$ belongs to $\mathbb{C}_{\geq}^{q \times q}$. Thus, $(A^\diamond, O_{q \times q}, \rho)$ coincides with the Nevanlinna parametrization (A, B, ν) of the Herglotz-Nevanlinna function $\mathrm{Rstr}_{\Pi_+} f$. In particular, ρ is exactly the (unique) Nevanlinna measure ν of $\mathrm{Rstr}_{\Pi_+} f$. For each $B \in \mathfrak{B}_{[\alpha,+\infty)}$, we have $\tilde{T}_{-\alpha}(B) \in \mathfrak{B}_{[0,+\infty)}$ and hence

$$\mu(B) = \left[\tilde{T}_{-\alpha}(\mu) \right] \left(\tilde{T}_{-\alpha}(B) \right) = \int_{\tilde{T}_{-\alpha}(B)} \left(\frac{1+x^2}{1+x} \cdot \frac{1+x}{1+x^2} \right) \left[\tilde{T}_{-\alpha}(\mu) \right](\mathrm{d}x)$$

$$= \int_{\tilde{T}_{-\alpha}(B)} \frac{1+x^2}{1+x} \rho(\mathrm{d}x) = \int_{\tilde{T}_{-\alpha}(B)} \frac{1+x^2}{1+x} \nu(\mathrm{d}x).$$

In particular, μ is uniquely determined.

(b) Let $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and $\mu \in \mathcal{M}_{\geq}^q([\alpha,+\infty))$ be such that F can be represented via (3.5) for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Then the mapping $\hat{\rho} \colon \mathfrak{B}_{[0,+\infty)} \to \mathbb{C}_{\geq}^{q \times q}$ given by $\hat{\rho}(B) := \int_B (1+x)/(1+x^2)[\tilde{T}_{-\alpha}(\mu)](\mathrm{d}x)$ is well defined and belongs to $\mathcal{M}_{\geq}^q([0,+\infty))$. Furthermore, f satisfies (3.9) with $\hat{\rho}$ instead of ρ for each $w \in \mathbb{C} \setminus [0,+\infty)$. Hence, using a result on holomorphic dependence of an integral on a complex parameter (see, e.g. [9, Ch. IV, §5, Satz 5.8]), we conclude that f is a matrix-valued function which is holomorphic in $\mathbb{C} \setminus [0,+\infty)$. Because of $F(z) = f(z-\alpha)$ for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$, we obtain then that F is holomorphic in $\mathbb{C} \setminus [\alpha,+\infty)$. From Lem. 3.5 we get $\mathrm{Im}\, F(w) \in \mathbb{C}_{\geq}^{q \times q}$ for each $w \in \Pi_+$ and $F(x) \in \mathbb{C}_{\geq}^{q \times q}$ for each $x \in (-\infty, \alpha)$. Thus, F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$. $\square$

Remark 3.7. In the following, if $F \in \mathcal{S}_{q;[\alpha,+\infty)}$ is given, then we will write (γ_F, μ_F) for the unique pair $(\gamma, \mu) \in \mathbb{C}_{\geq}^{q \times q} \times \mathcal{M}_{\geq}^q([\alpha,+\infty))$ which realizes the integral representation (3.5) for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$.

Example 3.8. Let $\alpha \in \mathbb{R}$ and let $A, B \in \mathbb{C}_{\geq}^{q \times q}$. Let $F \in \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ be the function from $\mathcal{S}_{q;[\alpha,+\infty)}$ which is defined in Example 2.2. Then $\gamma_F = A$ and $\mu_F = \delta_\alpha B$ where δ_α denotes the Dirac measure on $([\alpha,+\infty), \mathfrak{B}_{[\alpha,+\infty)})$ with unit mass at α.

Example 3.9. Let $\alpha \in \mathbb{R}$, let $\gamma \in \mathbb{C}_{\geq}^{q \times q}$, and let $F \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ be defined for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$ by $F(z) := \gamma$. In view of Thm. 3.6, then $F \in \mathcal{S}_{q;[\alpha,+\infty)}$, $\gamma_F = \gamma$, and μ_F is the zero measure belonging to $\mathcal{M}_{\geq}^q([\alpha,+\infty))$.

Now we state some observations on the arithmetic of the class $\mathcal{S}_{q;[\alpha,+\infty)}$.

Remark 3.10. If $F \in \mathcal{S}_{q;[\alpha,+\infty)}$ then $F^{\mathrm{T}} \in \mathcal{S}_{q;[\alpha,+\infty)}$ and $(\gamma_{F^{\mathrm{T}}}, \mu_{F^{\mathrm{T}}}) = (\gamma_F^{\mathrm{T}}, \mu_F^{\mathrm{T}})$.

Remark 3.11. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(q_k)_{k=1}^n$ be a sequence of positive integers. For each $k \in \mathbb{Z}_{1,n}$, let $F_k \in \mathcal{S}_{q_k;[\alpha,+\infty)}$. Then $F := \mathrm{diag}[F_k]_{k=1}^n$ belongs to $\mathcal{S}_{\sum_{k=1}^n q_k;[\alpha,+\infty)}$ and $(\gamma_F, \mu_F) = (\mathrm{diag}[\gamma_{F_k}]_{k=1}^n, \mathrm{diag}[\mu_{F_k}]_{k=1}^n)$. Moreover, if $A_k \in \mathbb{C}^{q_k \times q}$ for each $k \in \mathbb{Z}_{1,n}$, then $G := \sum_{k=1}^n A_k^* F_k A_k$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ and $(\gamma_G, \mu_G) = (\sum_{k=1}^n A_k^* \gamma_{F_k} A_k, \sum_{k=1}^n A_k^* \mu_{F_k} A_k)$.

Remark 3.12. Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. For each matrix $A \in \mathbb{C}^{q \times q}$ for which the matrix $\gamma_F + A$ is non-negative Hermitian, from Thm. 3.6 one can easily see that the function $G := F + A$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ and that $(\gamma_G, \mu_G) = (\gamma_F + A, \mu_F)$.

Proposition 3.13. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. Then $\lim_{y \to +\infty} F(\mathrm{i}y) = \gamma_F$.*

Proof. Let $(y_n)_{n=1}^{\infty}$ be a sequence from $[1, +\infty)$ such that $\lim_{n \to +\infty} y_n = +\infty$. Obviously, in view of Rem. 3.3, we have $\lim_{n \to +\infty} h_{\alpha,\mathrm{i}y_n}(t) = 0$ for each $t \in [\alpha, +\infty)$. Furthermore, for each $t \in [\alpha, +\infty)$, we get $|h_{\alpha,\mathrm{i}y_n}(t)| \leq 3 + |\alpha|$. By virtue of Prop. A.6, we obtain then $\lim_{n \to +\infty} \int_{[\alpha,+\infty)} h_{\alpha,\mathrm{i}y_n} \mathrm{d}\mu_F = O_{q \times q}$. Application of the integral representation stated in Thm. 3.6(a) completes the proof. $\qquad\square$

Corollary 3.14. *Let $\alpha \in \mathbb{R}$. Then $\mathcal{S}_{q;[\alpha,+\infty)}^{\diamond} = \{F \in \mathcal{S}_{q;[\alpha,+\infty)} \colon \gamma_F = O_{q \times q}\}$.*

Proof. Combine Prop. 3.13 and Rem. 3.7. $\qquad\square$

If $\mathcal{X}$ is a non-empty subset of $\mathbb{C}^q$, then we will use $\mathcal{X}^{\perp}$ to denote the orthogonal space of $\mathcal{X}$. For each $A \in \mathbb{C}^{p \times q}$, let $\mathcal{N}(A)$ be the null space of A and let $\mathcal{R}(A)$ be the column space of A.

Recall that a complex $q \times q$ matrix A is called an *EP matrix* if $\mathcal{R}(A^*) = \mathcal{R}(A)$. The class $\mathbb{C}_{\mathrm{EP}}^{q \times q}$ of these complex $q \times q$ matrices was introduced by Schwerdtfeger [20]. For a comprehensive treatment of the class $\mathbb{C}_{\mathrm{EP}}^{q \times q}$ against to the background of this paper, we refer the reader to [10, Appendix A]. If $G \in \mathcal{R}_q(\Pi_+)$ then it was proved in [10, Lem. 9.1] that, for each $w \in \Pi_+$, the matrix $G(w)$ belongs to $\mathbb{C}_{\mathrm{EP}}^{q \times q}$. Part (b) of the following proposition shows that an analogous result is true for functions belonging to the class $\mathcal{S}_{q;[\alpha,+\infty)}$. Furthermore, the following proposition contains also extensions to the class $\mathcal{S}_{q;[\alpha,+\infty)}$ of former results (see [10, Thm. 9.4], [11, Prop. 3.7]) concerning the class $\mathcal{R}_q(\Pi_+)$.

Proposition 3.15. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. Then:*

(a) *Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Then $\overline{z} \in \mathbb{C} \setminus [\alpha, +\infty)$ and $F^*(z) = F(\overline{z})$.*

(b) *For each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, the equations*

$$\mathcal{N}(F(z)) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, +\infty))) \tag{3.10}$$

and

$$\mathcal{R}(F(z)) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty))) \tag{3.11}$$

are valid. In particular, $\mathcal{N}(F(z)) = \mathcal{N}(F^*(z)) = [\mathcal{R}(F(z))]^{\perp}$ and $\mathcal{R}(F(z)) = \mathcal{R}(F^*(z)) = [\mathcal{N}(F(z))]^{\perp}$ for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

(c) *Let $r \in \mathbb{N}_0$. Then the following statements are equivalent:*

    (i) *For each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, the equation $\operatorname{rank} F(z) = r$ holds.*

    (ii) *There is some $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$ such that $\operatorname{rank} F(z_0) = r$.*

    (iii) *$\dim[\mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, +\infty)))] = r$.*

(d) *If $\det \gamma_F \neq 0$ or $\det[\mu_F([\alpha, +\infty))] \neq 0$ then $\det[F(w)] \neq 0$ for all $w \in \mathbb{C} \setminus [\alpha, +\infty)$.*

Proof. In view of Rem. 3.7, we have

$$\gamma_F \in \mathbb{C}_\geq^{q \times q} \qquad \text{and} \qquad \mu_F \in \mathcal{M}_\geq^q([\alpha, +\infty)). \qquad (3.12)$$

Taking into account Rem. 3.3, for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, we get then that $h_{\alpha,z}$ belongs to $\mathcal{L}^1([\alpha, +\infty), \mathfrak{B}_{[\alpha,+\infty)}, \mu_F; \mathbb{C})$.

(a) Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. From Rem. 3.7 and (3.12) we obtain

$$F^*(z) = \gamma_F^* + \left[\int_{[\alpha,+\infty)} \frac{1+t-\alpha}{t-z} \mu_F(\mathrm{d}t) \right]^* = \gamma_F + \int_{[\alpha,+\infty)} \frac{1+t-\alpha}{t-\overline{z}} \mu_F(\mathrm{d}t) = F(\overline{z}).$$

(b) Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. For each $t \in [\alpha, +\infty)$, we get then

$$\operatorname{Re} h_{\alpha,z}(t) = \frac{(t - \operatorname{Re} z)(1+t-\alpha)}{|t-z|^2} \qquad (3.13)$$

and

$$\operatorname{Im} h_{\alpha,z}(t) = \frac{(\operatorname{Im} z)(1+t-\alpha)}{|t-z|^2}. \qquad (3.14)$$

Since $h_{\alpha,z}$ belongs to $\mathcal{L}^1([\alpha, +\infty), \mathfrak{B}_{[\alpha,+\infty)}, \mu_F; \mathbb{C})$, from Lem. A.4(a) we see that

$$\mathcal{N}(\mu_F([\alpha, +\infty))) \subseteq \mathcal{N}\left(\int_{[\alpha,+\infty)} h_{\alpha,z} \mathrm{d}\mu_F \right) \qquad (3.15)$$

holds true. Now we consider an arbitrary $u \in \mathcal{N}(F(z))$. In view of the definition of the pair (γ_F, μ_F) (see Rem. 3.7), we have

$$u^* \gamma_F u + \int_{[\alpha,+\infty)} h_{\alpha,z} \mathrm{d}(u^* \mu_F u). = u^* F(z) u = 0. \qquad (3.16)$$

Consequently, because of (3.16), (3.12), and (3.14), then

$$0 = u^* \gamma_F u + \int_{[\alpha,+\infty)} \operatorname{Re} h_{\alpha,z} \mathrm{d}(u^* \mu_F u) \geq \int_{[\alpha,+\infty)} \operatorname{Re} h_{\alpha,z} \mathrm{d}(u^* \mu_F u) \qquad (3.17)$$

and

$$0 = \operatorname{Im}\left[\int_{[\alpha,+\infty)} h_{\alpha,z} \mathrm{d}(u^* \mu_F u) \right] = \int_{[\alpha,+\infty)} \frac{(\operatorname{Im} z)(1+t-\alpha)}{|t-z|^2} (u^* \mu_F u)(\mathrm{d}t) \qquad (3.18)$$

follow. In the case $\operatorname{Im} z \neq 0$, from (3.18) and (3.12) we get $(u^* \mu_F u)([\alpha, +\infty)) = 0$. If $z \in (-\infty, \alpha)$, then from (3.13) we see that $\operatorname{Re} h_{\alpha,z}(t) \in (0, +\infty)$ holds for each $t \in [\alpha, +\infty)$, and, by virtue of (3.17) and (3.12), we obtain $(u^* \mu_F u)([\alpha, +\infty)) = 0$. Thus $(u^* \mu_F u)([\alpha, +\infty)) = 0$ is proved in each case, which, in view of (3.12), implies $u \in$

$\mathcal{N}(\mu_F([\alpha,+\infty)))$. Taking into account a standard argument of the integration theory of non-negative Hermitian measures and (3.15), we conclude that

$$\int_{[\alpha,+\infty)} h_{\alpha,z}\mathrm{d}(u^*\mu_F u) = u^*\left(\int_{[\alpha,+\infty)} h_{\alpha,z}\mathrm{d}\mu_F\right)u = u^* \cdot O_{q\times 1} = 0.$$

Consequently, from (3.16) we infer $u^*\gamma_F u = 0$. Thus, (3.12) shows that u belongs to $\mathcal{N}(\gamma_F)$. Hence,

$$\mathcal{N}(F(z)) \subseteq \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha,+\infty))) \tag{3.19}$$

is valid. Now we are going to check that

$$\mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha,+\infty))) \subseteq \mathcal{N}(F(z)) \tag{3.20}$$

holds. For this reason, we consider an arbitrary $u \in \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha,+\infty)))$. From (3.15) we get then $F(z)u = \gamma_F u + (\int_{[\alpha,+\infty)} h_{\alpha,z}\mathrm{d}\mu_F)u = O_{q\times 1}$ and therefore $u \in \mathcal{N}(F(z))$. Hence (3.20) is verified. From (3.19) and (3.20) then (3.10) follows. Keeping in mind (a), (3.10) for $\overline{z}$ instead of z, and (3.12), standard arguments of functional analysis yield then

$$\begin{aligned}
\mathcal{R}(F(z)) &= [\mathcal{N}(F(z)^*)]^\perp = [\mathcal{N}(F(\overline{z}))]^\perp \\
&= [\mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha,+\infty)))]^\perp = \left[\mathcal{R}(\gamma_F)^\perp \cap \mathcal{R}(\mu_F([\alpha,+\infty)))^\perp\right]^\perp \\
&= \left([\mathrm{span}(\mathcal{R}(\gamma_F) \cup \mathcal{R}(\mu_F([\alpha,+\infty))))]^\perp\right)^\perp = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha,+\infty))).
\end{aligned}$$

Thus, (3.11) is proved. Using (3.10) for z and for $\overline{z}$ instead of z, from (a) we obtain $\mathcal{N}(F(z)) = \mathcal{N}(F^*(z)) = [\mathcal{R}(F(z))]^\perp$. Similarly, $\mathcal{R}(F(z)) = \mathcal{R}(F^*(z)) = [\mathcal{N}(F(z))]^\perp$ follows from (3.11) and (a).

(c)–(d) These are immediate consequences of (b). $\qquad\square$

Prop. 3.15 yields a generalization of a result due to Kats and Krein [17, Cor. 5.1]:

Corollary 3.16. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$, and let $z_0 \in \mathbb{C} \setminus [\alpha,+\infty)$. Then $F(z_0) = O_{q\times q}$ if and only if $F(z) = O_{q\times q}$ for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$.*

Proof. This is an immediate consequence of Prop. 3.15(c). $\qquad\square$

Corollary 3.17. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$, and let $\lambda \in \mathbb{R}$ be such that the matrix $\gamma_F - \lambda I_q$ is non-negative Hermitian. Then*

$$\mathcal{R}(F(z) - \lambda I_q) = \mathcal{R}(F(w) - \lambda I_q) \quad \text{and} \quad \mathcal{N}(F(z) - \lambda I_q) = \mathcal{N}(F(w) - \lambda I_q) \tag{3.21}$$

for all $z, w \in \mathbb{C} \setminus [\alpha,+\infty)$. In particular, if $\lambda \leq 0$, then λ is an eigenvalue of the matrix $F(z_0)$ for some $z_0 \in \mathbb{C} \setminus [\alpha,+\infty)$ if and only if λ is an eigenvalue of the matrix $F(z)$ for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. In this case, the eigenspaces $\mathcal{N}(F(z) - \lambda I_q)$ are independent of $z \in \mathbb{C} \setminus [\alpha,+\infty)$.

Proof. In view of Rem. 3.7 and Thm. 3.6, we conclude that the function $G\colon \mathbb{C}\backslash[\alpha,+\infty) \to \mathbb{C}^{q\times q}$ defined by $G(z) := F(z) - \lambda I_q$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$. The application of Prop. 3.15(b) to the function G yields then (3.21). Since the matrix γ_F is non-negative Hermitian, we have $\gamma_F - \lambda I_q \in \mathbb{C}^{q\times q}_{\geq}$ if $\lambda \leq 0$. Thus, the remaining assertions are an immediate consequence of (3.21). $\square$

At the end of this section we add a useful technical result.

Lemma 3.18. *Let* $\alpha \in \mathbb{R}$, *let* $A \in \mathbb{C}^{p\times q}$, *and let* $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. *Then the following statements are equivalent:*

(i) $\mathcal{N}(A) \subseteq \mathcal{N}(F(z))$ *for all* $z \in \mathbb{C} \setminus [\alpha,+\infty)$.

(ii) *There is a* $z_0 \in \mathbb{C} \setminus [\alpha,+\infty)$ *such that* $\mathcal{N}(A) \subseteq \mathcal{N}(F(z_0))$.

(iii) $\mathcal{N}(A) \subseteq \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha,+\infty)))$.

(iv) $FA^{\dagger}A = F$.

(v) $[\mathcal{N}(A)]^{\perp} \supseteq \mathcal{R}(F(z))$ *for all* $z \in \mathbb{C} \setminus [\alpha,+\infty)$.

(vi) *There is a* $z_0 \in \mathbb{C} \setminus [\alpha,+\infty)$ *such that* $[\mathcal{N}(A)]^{\perp} \supseteq \mathcal{R}(F(z_0))$.

(vii) $[\mathcal{N}(A)]^{\perp} \supseteq \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha,+\infty)))$.

(viii) $A^{\dagger}AF = F$.

Proof. "(i)$\Rightarrow$(ii)" and "(v)$\Rightarrow$(vi)": These implications hold true obviously.

"(i)$\Leftrightarrow$(iii)" and "(ii)$\Rightarrow$(iii)": Use equation (3.10) in Prop. 3.15(b).

"(i)$\Leftrightarrow$(iv)": This equivalence follows from a well-known result for the Moore-Penrose inverse of complex matrices.

"(i)$\Leftrightarrow$(v)": Because of Prop. 3.15(b), we have $\mathcal{N}(F(z)) = \mathcal{R}(F(z))^{\perp}$ for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Hence, (i) and (v) are equivalent.

"(v)$\Leftrightarrow$(vii)" and "(vi)$\Rightarrow$(vii)": Use equation (3.11) in Prop. 3.15(b).

"(v)$\Leftrightarrow$(viii)": Use $\mathcal{N}(A)^{\perp} = \mathcal{R}(A^*)$ and $A^{\dagger}A\mathcal{R}(A^*) = \mathcal{R}(A^*)$. $\square$

Now we apply the preceding results to the subclass $\mathcal{S}^{\diamond}_{q;[\alpha,+\infty)}$ of $\mathcal{S}_{q;[\alpha,+\infty)}$ (see Notation 2.9).

Example 3.19. Let $\alpha \in \mathbb{R}$ and let $F\colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q\times q}$ be defined by $F(z) := O_{q\times q}$. In view of Example 3.9 and Cor. 3.14, one can easily see then that F belongs to $\mathcal{S}^{\diamond}_{q;[\alpha,+\infty)}$ and that μ_F is the zero measure belonging to $\mathcal{M}^q_{\geq}([\alpha,+\infty))$.

Remark 3.20. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(q_k)^n_{k=1}$ be a sequence of positive integers. For each $k \in \mathbb{Z}_{1,n}$, let $F_k \in \mathcal{S}^{\diamond}_{q_k;[\alpha,+\infty)}$ and let $A_k \in \mathbb{C}^{q_k\times q}$. In view of Cor. 3.14 and Rem. 3.11, then:

(a) The function $G := \sum^n_{k=1} A^*_k F_k A_k$ belongs to $\mathcal{S}^{\diamond}_{q;[\alpha,+\infty)}$ and $\mu_G = \sum^n_{k=1} A^*_k \mu_{F_k} A_k$.

(b) The function $F := \mathrm{diag}[F_k]^n_{k=1}$ belongs to $\mathcal{S}^{\diamond}_{\sum^n_{k=1} q_k;[\alpha,+\infty)}$ and $\mu_F = \mathrm{diag}[\mu_{F_k}]^n_{k=1}$.

Remark 3.21. Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{\diamond}$. In view of Cor. 3.14 and Prop. 3.15(b), then $\mathcal{N}(F(z)) = \mathcal{N}(\mu_F([\alpha,+\infty)))$ and $\mathcal{R}(F(z)) = \mathcal{R}(\mu_F([\alpha,+\infty)))$ for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$.

4. Characterizations of the class $\mathcal{S}_{q;[\alpha,+\infty)}$

In this section, we give several characterizations of the class $\mathcal{S}_{q;[\alpha,+\infty)}$.

Lemma 4.1. *Let $\alpha \in \mathbb{R}$, let $F \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ be holomorphic, and let $F^{\square} \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ be defined by*

$$F^{\square}(z) := (z - \alpha)F(z). \tag{4.1}$$

Suppose that $\mathrm{Rstr}_{\Pi_+} F$ and $\mathrm{Rstr}_{\Pi_+} F^{\square}$ both belong to $\mathcal{R}_q(\Pi_+)$. Then $(\mathrm{Re}\, F)((-\infty,\alpha)) \subseteq \mathbb{C}_{\geq}^{q \times q}$.

Proof. We consider an arbitrary $x \in (-\infty,\alpha)$. For each $n \in \mathbb{N}$, we have then

$$\mathrm{Re}\, F\left(x + \frac{\mathrm{i}}{n}\right) = n\, \mathrm{Im}\, F^{\square}\left(x + \frac{\mathrm{i}}{n}\right) + n(\alpha - x)\, \mathrm{Im}\, F\left(x + \frac{\mathrm{i}}{n}\right). \tag{4.2}$$

For each $n \in \mathbb{N}$, $\mathrm{Rstr}_{\Pi_+} F^{\square} \in \mathcal{R}_q(\Pi_+)$ implies $n\, \mathrm{Im}\, F^{\square}(x + \mathrm{i}/n) \in \mathbb{C}_{\geq}^{q \times q}$, whereas $\mathrm{Rstr}_{\Pi_+} F \in \mathcal{R}_q(\Pi_+)$ yields $n(\alpha - x)\, \mathrm{Im}\, F(x + \mathrm{i}/n) \in \mathbb{C}_{\geq}^{q \times q}$. Thus, (4.2) provides us $\mathrm{Re}\, F(x + \mathrm{i}/n) \in \mathbb{C}_{\geq}^{q \times q}$ for each $n \in \mathbb{N}$. Since F is continuous, we get then $\mathrm{Re}\, F(x) = \lim_{n \to \infty} \mathrm{Re}\, F(x + \mathrm{i}/n)$. In particular, the matrix $\mathrm{Re}\, F(x)$ is non-negative Hermitian. $\square$

In order to give further characterizations of the class $\mathcal{S}_{q;[\alpha,+\infty)}$, we state the following technical result. The proof of which uses an idea which is originated in [19, Thm. A.5].

Lemma 4.2. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. Then $F^{\square} \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ defined by (4.1) is holomorphic and fulfills*

$$\mathrm{Im}\, F^{\square}(z) = (\mathrm{Im}\, z)\left[\gamma_F + \int_{[\alpha,+\infty)} \frac{(1 + t - \alpha)(t - \alpha)}{|t - z|^2} \mu_F(\mathrm{d}t)\right] \tag{4.3}$$

for each $z \in \mathbb{C} \setminus [\alpha,+\infty)$. Furthermore, $\frac{1}{\mathrm{Im}\, z} \mathrm{Im}\, F^{\square}(z) \in \mathbb{C}_{\geq}^{q \times q}$ for each $z \in \mathbb{C} \setminus \mathbb{R}$ and $F^{\square}(x) \in \mathbb{C}_{\mathrm{H}}^{q \times q}$ for each $x \in (-\infty,\alpha)$.

Proof. Since F is holomorphic, the matrix-valued function $F^{\square}$ is holomorphic as well. In view of Rem. 3.7, using a well-known result on integrals with respect to non-negative Hermitian measures, we have

$$[F(z)]^* = \gamma_F + \int_{[\alpha,+\infty)} \frac{1 + t - \alpha}{t - \overline{z}} \mu_F(\mathrm{d}t) \tag{4.4}$$

for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Thus, from Rem. 3.7 and (4.4) we get

$$2\mathrm{i} \operatorname{Im} F^{\square}(z) = (z - \alpha)F(z) - (\overline{z} - \alpha)[F(z)]^*$$

$$= (z - \overline{z})\gamma_F + \int_{[\alpha,+\infty)} (1 + t - \alpha)\left(\frac{z - \alpha}{t - z} - \frac{\overline{z} - \alpha}{t - \overline{z}}\right)\mu_F(\mathrm{d}t) \qquad (4.5)$$

for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Since $(z-\alpha)/(t-z) - (\overline{z}-\alpha)/(t-\overline{z}) = 2\mathrm{i}(\operatorname{Im} z)(t-\alpha)/|t-z|^2$ holds for every choice of $z \in \mathbb{C} \setminus [\alpha, +\infty)$ and $t \in [\alpha, +\infty)$, from (4.5) it follows (4.3) for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Since $(1 + t - \alpha)(t - \alpha)/|t - z|^2 \in [0, +\infty)$ holds true for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$ and each $t \in [\alpha, +\infty)$, from $\gamma_F \in \mathbb{C}_{\geq}^{q \times q}$ and (4.3) we get $\frac{1}{\operatorname{Im} z} \operatorname{Im} F^{\square}(z) \in \mathbb{C}_{\geq}^{q \times q}$ for each $z \in \mathbb{C} \setminus \mathbb{R}$. In view of Lem. 3.4, then $F^{\square}(x) = \operatorname{Re} F^{\square}(x)$ and hence $F^{\square}(x) \in \mathbb{C}_{\mathrm{H}}^{q \times q}$ for each $x \in (-\infty, \alpha)$. $\square$

Proposition 4.3. *Let $\alpha \in \mathbb{R}$, let $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ be a matrix-valued function, and let $F^{\square} \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ be defined by (4.1). Then F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ if and only if the following two conditions hold true:*

(I) F is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$.

(II) The matrix-valued functions $\operatorname{Rstr}_{\Pi_+} F$ and $\operatorname{Rstr}_{\Pi_+} F^{\square}$ both belong to $\mathcal{R}_q(\Pi_+)$.

Proof. First suppose that F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$. Then (I) and $\operatorname{Rstr}_{\Pi_+} F \in \mathcal{R}_q(\Pi_+)$ follow from the definition of the class $\mathcal{S}_{q;[\alpha,+\infty)}$. Furthermore, Lem. 4.2 provides us $\operatorname{Rstr}_{\Pi_+} F^{\square} \in \mathcal{R}_q(\Pi_+)$.

Conversely, now suppose that (I) and (II) hold true. Because of the definition of the classes $\mathcal{R}_q(\Pi_+)$ and $\mathcal{S}_{q;[\alpha,+\infty)}$, it then remains to prove that $F((-\infty, \alpha)) \subseteq \mathbb{C}_{\geq}^{q \times q}$. We consider an arbitrary $x \in (-\infty, \alpha)$. First we show that $\operatorname{Im} F(x) = O_{q \times q}$. Because of (II), for each $n \in \mathbb{N}$, the matrices $\operatorname{Im} F(x + \mathrm{i}/n)$ and $\operatorname{Im} F^{\square}(x + \mathrm{i}/n)$ are non-negative Hermitian. Thus, the matrices $\operatorname{Im} F(x)$ and $\operatorname{Im} F^{\square}(x)$ are (as limits of the sequences $(\operatorname{Im} F(x + \mathrm{i}/n))_{n=1}^{\infty}$ and $(\operatorname{Im} F^{\square}(x + \mathrm{i}/n))_{n=1}^{\infty}$, respectively) non-negative Hermitian as well. Since (4.1) implies $\operatorname{Im} F^{\square}(x) = (x - \alpha) \operatorname{Im} F(x)$, we get then $-\operatorname{Im} F(x) = \frac{1}{\alpha - x} \operatorname{Im} F^{\square}(x) \in \mathbb{C}_{\geq}^{q \times q}$, which together with $\operatorname{Im} F(x) \in \mathbb{C}_{\geq}^{q \times q}$ yields $\operatorname{Im} F(x) = O_{q \times q}$. Hence, $\operatorname{Re} F(x) = F(x)$. Because of (I), (II), and Lem. 4.1, we have $\operatorname{Re} F(x) \in \mathbb{C}_{\geq}^{q \times q}$. Thus, $F((-\infty, \alpha)) \subseteq \mathbb{C}_{\geq}^{q \times q}$. $\square$

Let $\mathbb{C}_{\alpha,-} := \{z \in \mathbb{C} \colon \operatorname{Re} z \in (-\infty, \alpha)\}$.

Proposition 4.4. *Let $\alpha \in \mathbb{R}$ and let $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ be a matrix-valued function. Then F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ if and only if the following four conditions are fulfilled:*

(I) F is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$.

(II) For each $z \in \Pi_+$, the matrix $\operatorname{Im} F(z)$ is non-negative Hermitian.

(III) For each $z \in \Pi_-$, the matrix $-\operatorname{Im} F(z)$ is non-negative Hermitian.

(IV) For each $z \in \mathbb{C}_{\alpha,-}$, the matrix $\operatorname{Re} F(z)$ is non-negative Hermitian.

Proof. First suppose that $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. By definition of the class $\mathcal{S}_{q;[\alpha,+\infty)}$, conditions (I) and (II) are fulfilled. From Thm. 3.6 and Lem. 3.5 we obtain (III) and (IV).

Conversely, (I)–(III) and Lem. 3.4 imply $\operatorname{Im} F(x) = O_{q\times q}$ for all $x \in (-\infty, \alpha)$. In view of (IV), we have then $F((-\infty, \alpha)) \subseteq \mathbb{C}_\geq^{q\times q}$. Together with (I) and (II), this yields $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. $\qquad\square$

5. The class $\mathcal{S}_{0,q;[\alpha,+\infty)}$

In this section, we prove an important integral representation for functions which belong to the class $\mathcal{S}_{0,q;[\alpha,+\infty)}$. It can be considered as modified integral representation of the functions belonging to the class $\mathcal{R}_{0,q}(\Pi_+) := \{F \in \mathcal{R}_q(\Pi_+) \colon \sup_{y\in[1,+\infty)} y\|F(iy)\| < +\infty\}$ (see [4, Thm. 8.7]). Observe that if $\alpha \in \mathbb{R}$ and if $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then in view of Lem. A.8(a), for each $\sigma \in \mathcal{M}_\geq^q([\alpha, +\infty))$, the integral $\int_{[\alpha,+\infty)} 1/(t-z)\sigma(\mathrm{d}t)$ exists.

Theorem 5.1. *Let $\alpha \in \mathbb{R}$ and let $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q\times q}$. Then:*

(a) *If $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$, then there is a unique non-negative Hermitian measure $\sigma \in \mathcal{M}_\geq^q([\alpha, +\infty))$ such that*

$$F(z) = \int_{[\alpha,+\infty)} \frac{1}{t-z}\sigma(\mathrm{d}t) \tag{5.1}$$

for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

(b) *If there is a non-negative Hermitian measure $\sigma \in \mathcal{M}_\geq^q([\alpha, +\infty))$ such that F can be represented via (5.1) for each $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then F belongs to the class $\mathcal{S}_{0,q;[\alpha,+\infty)}$.*

Proof. We modify ideas of proofs of integral representations of similar classes of holomorphic functions (see [19, Appendix]).

(a) First suppose $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$. Then $F \in \mathcal{S}_{q;[\alpha,+\infty)}$ and the function $F := \operatorname{Rstr}_{\Pi_+} F$ belongs to the class $\mathcal{R}_{0,q}(\Pi_+)$. From a matricial version of a well-known integral representation of functions belonging to $\mathcal{R}_{0,1}(\Pi_+)$ (see, e.g. [4, Thm. 8.7]) we know that there is a unique $\mu \in \mathcal{M}_\geq^q(\mathbb{R})$ such that

$$F(w) = \int_\mathbb{R} \frac{1}{t-w}\mu(\mathrm{d}t) \tag{5.2}$$

for all $w \in \Pi_+$, namely the so-called spectral measure of F, i.e., for each $B \in \mathfrak{B}_\mathbb{R}$, we have $\mu(B) = \int_B (1+t^2)\nu(\mathrm{d}t)$, where ν denotes the Nevanlinna measure of F. Prop. 4.4 shows that F is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$ and that $\operatorname{Im} F(z) \in \mathbb{C}_\geq^{q\times q}$ for all $z \in \Pi_+$ and $-\operatorname{Im} F(z) \in \mathbb{C}_\geq^{q\times q}$ for all $z \in \Pi_-$. Hence, for each $t \in (-\infty, \alpha)$, we have $F^*(t) = F(t)$. Applying the Stieltjes-Perron inversion formula (see, e.g. [4, Thm. 8.2]), one can verify that $\nu((-\infty, \alpha)) = O_{q\times q}$. Hence $\mu((-\infty, \alpha)) = O_{q\times q}$. Consequently, formula (5.2) shows that (5.1) holds for each $z \in \Pi_+$, where $\sigma := \operatorname{Rstr}_{\mathfrak{B}_{[\alpha,+\infty)}} \mu$. Since $[\alpha, +\infty)$ is a closed

interval the function $G\colon \mathbb{C}\setminus[\alpha,+\infty) \to \mathbb{C}^{q\times q}$ defined by $G(z) := \int_{[\alpha,+\infty)} 1/(t-z)\sigma(\mathrm{d}t)$ is holomorphic (see, e. g. [9, Ch. IV, §5, Satz 5.8]). Because of $F(w) = G(w)$ for each $w \in \Pi_+$, we have $F = G$. If σ is an arbitrary measure belonging to $\mathcal{M}_{\geq}^q([\alpha,+\infty))$ such that (5.1) holds for each $z \in \mathbb{C}\setminus[\alpha,+\infty)$, then using standard arguments of measure theory and the uniqueness of the non-negative Hermitian $q \times q$ measure ν in the integral representation (2.2), one gets necessary $\sigma = \mathrm{Rstr}_{\mathfrak{B}_{[\alpha,+\infty)}}\mu$.

(b) Let $\sigma \in \mathcal{M}_{\geq}^q([\alpha,+\infty))$ such that (5.1) holds for each $z \in \mathbb{C}\setminus[\alpha,+\infty)$. Then F is holomorphic (see, e. g. [9, Ch. IV, §5, Satz 5.8]) and, for each $z \in \mathbb{C}\setminus\mathbb{R}$, we have

$$\frac{1}{\mathrm{Im}\, z}\,\mathrm{Im}\, F(z) = \int_{[\alpha,+\infty)} \frac{1}{\mathrm{Im}\, z}\,\mathrm{Im}\,\frac{1}{t-z}\sigma(\mathrm{d}t) = \int_{[\alpha,+\infty)} \frac{1}{|t-z|^2}\sigma(\mathrm{d}t) \in \mathbb{C}_{\geq}^{q\times q}.$$

and, for each z belonging to $\mathbb{C}_{\alpha,-}$, moreover

$$\mathrm{Re}\, F(z) = \int_{[\alpha,+\infty)} \mathrm{Re}\,\frac{1}{t-z}\sigma(\mathrm{d}t) = \int_{[\alpha,+\infty)} \frac{t-\mathrm{Re}\, z}{|t-z|^2}\sigma(\mathrm{d}t) \in \mathbb{C}_{\geq}^{q\times q}.$$

Thus, $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. From the definition of F and [4, Thm. 8.7(b)] we see that $\mathrm{Rstr}_{\Pi_+} F \in \mathcal{R}_{0,q}(\Pi_+)$, where $\mathcal{R}_{0,q}(\Pi_+)$ is the class of all $H \in \mathcal{R}_q(\Pi_+)$ satisfying $\sup_{y\in[1,+\infty)} y\|H(\mathrm{i}y)\|_{\mathrm{E}} < +\infty$. Thus, (2.1) is satisfied. Hence, $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$ holds. $\square$

If σ is a measure belonging to $\mathcal{M}_{\geq}^q([\alpha,+\infty))$, then we will call the matrix-valued function $F\colon \mathbb{C}\setminus[\alpha,+\infty) \to \mathbb{C}^{q\times q}$ which is, for each $z \in \mathbb{C}\setminus[\alpha,+\infty)$, given by (5.1) the $[\alpha,+\infty)$-*Stieltjes transform of* σ. If $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$, then the unique measure σ which belongs to $\mathcal{M}_{\geq}^q([\alpha,+\infty))$ and which fulfills (5.1) for each $z \in \mathbb{C}\setminus[\alpha,+\infty)$ is said to be the $[\alpha,+\infty)$-*Stieltjes measure of* F and will be denoted by σ_F.

Note that, in view of Thm. 5.1, the matricial Stieltjes moment problem $\mathsf{M}[[\alpha,+\infty);(s_j)_{j=0}^m,=]$ can be obviously reformulated in the language of $[\alpha,+\infty)$-Stieltjes transforms of non-negative Hermitian measures. We omit the details.

Remark 5.2. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(q_k)_{k=1}^n$ be a sequence of positive integers. For each $k \in \mathbb{Z}_{1,n}$, let $F_k \in \mathcal{S}_{0,q_k;[\alpha,+\infty)}$, and let σ_{F_k} be the $[\alpha,+\infty)$-Stieltjes measure of F_k. Then $F := \mathrm{diag}[F_k]_{k=1}^n$ belongs to $\mathcal{S}_{0,\sum_{k=1}^n q_k;[\alpha,+\infty)}$ and $\mathrm{diag}[\sigma_{F_k}]_{k=1}^n$ is the $[\alpha,+\infty)$-Stieltjes measure of F. Moreover, if $A_k \in \mathbb{C}^{q_k\times q}$ for each $k \in \mathbb{Z}_{1,n}$, then $G := \sum_{k=1}^n A_k^* F_k A_k$ belongs to $\mathcal{S}_{0,q;[\alpha,+\infty)}$ and $\sum_{k=1}^n A_k^* \sigma_{F_k} A_k$ is the $[\alpha,+\infty)$-Stieltjes measure of G.

Proposition 5.3. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$, and let σ_F be the $[\alpha,+\infty)$-Stieltjes measure of F. For each $z \in \mathbb{C}\setminus[\alpha,+\infty)$, then*

$$\mathcal{N}(F(z)) = \mathcal{N}(\sigma_F([\alpha,+\infty))) \qquad and \qquad \mathcal{R}(F(z)) = \mathcal{R}(\sigma_F([\alpha,+\infty))). \qquad (5.3)$$

Furthermore,

$$\sigma_F([\alpha,+\infty)) = -\mathrm{i}\lim_{y\to+\infty} yF(\mathrm{i}y). \qquad (5.4)$$

In particular, $\mathrm{rank}\, F(z) = \mathrm{rank}\, \sigma_F([\alpha,+\infty))$ *holds true for each* $z \in \mathbb{C}\setminus[\alpha,+\infty)$.

Proof. Let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. From Thm. 5.1(a) and Lem. A.8(b) we obtain the second equation in (5.3). The first one is an immediate consequence of the second one and the equation $F(z) = F^*(\overline{z})$, which can be seen from (5.1). Because of Thm. 5.1(a) and Lem. A.8(c), the equation (5.4) holds true. □

6. Moore-Penrose inverses of functions belonging to the class $\mathcal{S}_{q;[\alpha,+\infty)}$

We start with some further notation. If $\mathcal{Z}$ is a non-empty subset of $\mathbb{C}$ and if a matrix-valued function $F: \mathcal{Z} \to \mathbb{C}^{p \times q}$ is given, then let $F^\dagger: \mathcal{Z} \to \mathbb{C}^{q \times p}$ be defined by $F^\dagger(z) := [F(z)]^\dagger$, where $[F(z)]^\dagger$ stands for the Moore-Penrose inverse of the matrix $F(z)$. In [11] (see also [10]), we investigated the Moore-Penrose inverse of an arbitrary function F belonging to the class $\mathcal{R}_q(\Pi_+)$. In particular, it turned out that $-F^\dagger$ belongs to $\mathcal{R}_q(\Pi_+)$ (see [10, Thm. 9.4]). The close relation between $\mathcal{S}_{q;[\alpha,+\infty)}$ and $\mathcal{R}_q(\Pi_+)$ suggests now to study the Moore-Penrose inverse of a function $F \in \mathcal{S}_{q;[\alpha,+\infty)}$.

Lemma 6.1. *Let $\alpha \in \mathbb{R}$ and $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. Then $F^\dagger$ is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$.*

Proof. In view of formulas (3.10) and (3.11), we obtain for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$ the identities $\mathcal{N}(F(z)) = \mathcal{N}(F(\mathrm{i}))$ and $\mathcal{R}(F(z)) = \mathcal{R}(F(\mathrm{i}))$. Thus, the application of [15, Prop. 8.4] completes the proof. □

Let $\alpha \in \mathbb{R}$ and $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. Then Lem. 6.1 suggests to look if there are functions closely related to $F^\dagger$ which belong again to $\mathcal{S}_{q;[\alpha,+\infty)}$. Against to this background, we are led to the function $G: \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ defined by $G(z) := -(z - \alpha)^{-1}[F(z)]^\dagger$.

Remark 6.2. If $A \in \mathbb{C}_{\mathrm{EP}}^{q \times q}$, i.e., if $A \in \mathbb{C}^{q \times q}$ fulfills $\mathcal{R}(A^*) = \mathcal{R}(A)$, then $\mathrm{Im}(A^\dagger) = -A^\dagger (\mathrm{Im}\, A)(A^\dagger)^*$ (see also [10, Propositions A.5 and A.6]).

Theorem 6.3. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. Then $G: \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ defined by $G(z) := -(z - \alpha)^{-1}[F(z)]^\dagger$ belongs to the class $\mathcal{S}_{q;[\alpha,+\infty)}$ as well.*

Proof. Lem. 6.1 yields that the function $F^\dagger$ is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$. Consequently, the function G is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$. Let $z \in \Pi_+$. Using Prop. 3.15(b), we have $\mathcal{R}(F^*(z)) = \mathcal{R}(F(z))$. Hence, because of Rem. 6.2, the equations

$$\mathrm{Im}\, G(z) = G(z)(\mathrm{Im}[(z - \alpha)F(z)])G^*(z) \tag{6.1}$$

and

$$\mathrm{Im}[(z - \alpha)G(z)] = \mathrm{Im}\left[-F^\dagger(z)\right] = F^\dagger(z)[\mathrm{Im}\, F(z)]\left[F^\dagger(z)\right]^* \tag{6.2}$$

hold. Taking into account $F \in \mathcal{S}_{q;[\alpha,+\infty)}$, the application of Prop. 4.3 yields

$$\mathrm{Im}\, F(z) \in \mathbb{C}_{\geq}^{q \times q} \qquad \text{and} \qquad \mathrm{Im}[(z - \alpha)F(z)] \in \mathbb{C}_{\geq}^{q \times q}. \tag{6.3}$$

Thus, combining (6.1) (resp. (6.2)) and (6.3), we get $\mathrm{Im}\, G(z) \in \mathbb{C}_{\geq}^{q \times q}$ and $\mathrm{Im}[(z - \alpha)G(z)] \in \mathbb{C}_{\geq}^{q \times q}$. Now, the application of Prop. 4.3 yields $G \in \mathcal{S}_{q;[\alpha,+\infty)}$. □

Now we specify the result of Thm. 6.3 for functions belonging to $\mathcal{S}_{0,q;[\alpha,+\infty)}$.

Proposition 6.4. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$, and let σ_F be the $[\alpha,+\infty)$-Stieltjes measure of F. Then $G \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ defined by $G(z) := -(z-\alpha)^{-1}[F(z)]^\dagger$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ and*

$$\gamma_G = [\sigma_F([\alpha,+\infty))]^\dagger. \tag{6.4}$$

In particular, if F is not the constant function with value $O_{q \times q}$, then $G \in \mathcal{S}_{q;[\alpha,+\infty)} \setminus \mathcal{S}_{0,q;[\alpha,+\infty)}$.

Proof. In view of Thm. 6.3, we have $G \in \mathcal{S}_{q;[\alpha,+\infty)}$. From Prop. 3.13 we obtain

$$\gamma_G = \lim_{y \to +\infty} G(iy). \tag{6.5}$$

Since F belongs to $\mathcal{S}_{0,q;[\alpha,+\infty)}$, we have $\lim_{y\to+\infty} F(iy) = O_{q \times q}$. Prop. 5.3 yields (5.4). Consequently, $\lim_{y\to+\infty}(\alpha - iy)F(iy) = \sigma_F([\alpha,+\infty))$. In view of Prop. 5.3, we have $\mathcal{R}(F(iy)) = \mathcal{R}(\sigma_F([\alpha,+\infty)))$ and, in particular, $\mathrm{rank}[(\alpha - iy)F(iy)] = \mathrm{rank}\,\sigma_F([\alpha,+\infty))$ for each $y \in (0,+\infty)$. Hence, taking into account [3, Thm. 10.4.1], we obtain

$$\lim_{y \to +\infty}\left([(\alpha - iy)F(iy)]^\dagger\right) = [\sigma_F([\alpha,+\infty))]^\dagger. \tag{6.6}$$

Since $G(iy) = -(iy - \alpha)^{-1}F^\dagger(iy) = [(\alpha - iy)F(iy)]^\dagger$ holds true for each $y \in (0,+\infty)$, from (6.5) and (6.6) we get then (6.4). Now assume that G belongs to $\mathcal{S}_{0,q;[\alpha,+\infty)}$. From the definition of the class $\mathcal{S}_{0,q;[\alpha,+\infty)}$ we obtain then $\lim_{y\to+\infty} G(iy) = O_{q \times q}$, which, in view of (6.5) and (6.4), implies $\sigma_F([\alpha,+\infty)) = O_{q \times q}^\dagger = O_{q \times q}$. Prop. 5.3 yields then $\mathcal{N}(F(z)) = \mathbb{C}^q$ and hence $F(z) = O_{q \times q}$ for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. This proves $G \notin \mathcal{S}_{0,q;[\alpha,+\infty)}$ if F is not the constant function with value $O_{q \times q}$. $\square$

For the special choice $q = 1$ and $\alpha = 0$, the following class was introduced by Kats/Krein [17, Def. D1.5.2].

Notation 6.5. Let $\alpha \in \mathbb{R}$. Then let $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ be the class of all matrix-valued functions $F \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ which fulfill the following two conditions:

(I) F is holomorphic in $\mathbb{C} \setminus [\alpha,+\infty)$ with $\mathrm{Rstr}_{\Pi_+} F \in \mathcal{R}_q(\Pi_+)$.

(II) For all $x \in (-\infty,\alpha)$, the matrix $-F(x)$ is non-negative Hermitian.

Remark 6.6. Let $\alpha \in \mathbb{R}$, then $F \in \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ if and only if $u^* F u \in \mathcal{S}_{1;[\alpha,+\infty)}^{[-1]}$ for all $u \in \mathbb{C}^q$.

Example 6.7. Let $\alpha \in \mathbb{R}$ and let $D, E \in \mathbb{C}_{\geq}^{q \times q}$. Then $F \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -D + (z-\alpha)E$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$.

Lemma 6.8. *Let* $\alpha \in \mathbb{R}$ *and let* $f \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}$ *be such that there are real numbers* d *and* e *and a finite signed measure* ρ *on* $((\alpha, +\infty), \mathfrak{B}_{(\alpha,+\infty)})$ *such that* $f(z) = -d + (z - \alpha)[e + \int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z)\rho(\mathrm{d}t)]$ *holds true for all* $z \in \mathbb{C} \setminus [\alpha, +\infty)$. *Then* $d = -\lim_{x \to +0} f(\alpha - x)$ *and* $e = -\lim_{x \to +\infty}[f(\alpha - x) + d]/x$. *Furthermore,* d, e, *and* ρ *are uniquely determined.*

Proof. With $z_x := \alpha - x$ we have $(1 + t - \alpha)/(t - z_x) = (1 + t - \alpha)/(t - \alpha + x) \geq 0$ for all $t \in (\alpha, +\infty)$ and all $x \in (0, +\infty)$, which decreases to 0 as x increases to infinity. Since the signed measure ρ is finite, its Jordan decomposition $\rho = \rho_+ - \rho_-$ consists of two finite measures. Hence, $\int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z_1)\rho_\pm(\mathrm{d}t) = \rho_\pm((\alpha, +\infty)) < \infty$ holds true. Thus, Lebesgue's monotone convergence theorem yields $\lim_{x \to +\infty} \int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z_x)\rho_\pm(\mathrm{d}t) = 0$, which implies $\lim_{x \to +\infty}[f(z_x) + d]/(z_x - \alpha) = e$. Furthermore, we have $-(z_x - \alpha)(1 + t - \alpha)/(t - z_x) = (1 + t - \alpha)/[1 + (t - \alpha)/x] \geq 0$ for all $t \in (\alpha, +\infty)$ and all $x \in (0, +\infty)$, which decreases to 0 as x decreases to 0. Since $\int_{(\alpha,+\infty)}[-(z_1 - \alpha)(1 + t - \alpha)/(t - z_1)]\rho_\pm(\mathrm{d}t) = \rho_\pm((\alpha, +\infty)) < \infty$ holds true, Lebesgue's monotone convergence theorem yields $\lim_{x \to +0} \int_{(\alpha,+\infty)}[-(z_x - \alpha)(1 + t - \alpha)/(t - z_x)]\rho_\pm(\mathrm{d}t) = 0$ showing $-\lim_{x \to +0} f(z_x) = d$. In particular, d and e are uniquely determined. Now let σ be an arbitrary finite signed measure on $((\alpha, +\infty), \mathfrak{B}_{(\alpha,+\infty)})$ such that $f(z) = -d + (z - \alpha)[e + \int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z)\sigma(\mathrm{d}t)]$ holds true for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Then $\int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z)\sigma(\mathrm{d}t) = \int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z)\rho(\mathrm{d}t)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Since the signed measure σ is finite, its Jordan decomposition $\sigma = \sigma_+ - \sigma_-$ consists of two finite measures. Hence, we obtain $\int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z)(\sigma_+ + \rho_-)(\mathrm{d}t) = \int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z)(\rho_+ + \sigma_-)(\mathrm{d}t)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$ with finite measures $\sigma_+ + \rho_-$ and $\rho_+ + \sigma_-$. Using Thm. 3.6, it is readily checked then that $\sigma_+ + \rho_-$ and $\rho_+ + \sigma_-$ coincide. Consequently, $\sigma = \rho$ follows. $\qquad\square$

Lemma 6.9. *Let* $\alpha \in \mathbb{R}$ *and let* $f \in \mathcal{S}_{1;[\alpha,+\infty)}^{[-1]}$. *Then there are unique non-negative real numbers* d *and* e *and a unique measure* $\rho \in \mathcal{M}_{\geq}^1((\alpha, +\infty))$ *such that* $f(z) = -d + (z - \alpha)[e + \int_{(\alpha,+\infty)}(1 + t - \alpha)/(t - z)\rho(\mathrm{d}t)]$ *holds true for all* $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

Proof. Obviously, the function $g \colon \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}$ defined by $g(w) := f(w + \alpha)$ belongs to $\mathcal{S}_{1;[0,+\infty)}^{[-1]}$. Hence, by virtue of [17, Thm. S1.5.2], there exist unique numbers $a \in (-\infty, 0]$ and $b \in [0, +\infty)$ and a unique measure τ on $((0, +\infty), \mathfrak{B}_{(0,+\infty)})$ with $\int_{(0,+\infty)} 1/(x + x^2)\tau(\mathrm{d}x) < \infty$ such that $g(w) = a + bw + \int_{(0,+\infty)}[1/(x - w) - 1/x]\tau(\mathrm{d}x)$ for all $w \in \mathbb{C} \setminus [0, +\infty)$. Then χ defined on $\mathfrak{B}_{(0,+\infty)}$ by $\chi(B) := \int_B 1/(x + x^2)\tau(\mathrm{d}x)$ is a finite measure on $((0, +\infty), \mathfrak{B}_{(0,+\infty)})$ and the integral $\int_{(0,+\infty)}(x + x^2)[1/(x - w) - 1/x]\chi(\mathrm{d}x)$ exists for all $w \in \mathbb{C} \setminus [0, +\infty)$ and equals to $\int_{(0,+\infty)}[1/(x - w) - 1/x]\tau(\mathrm{d}x)$. We have $(x + x^2)[1/(x - w) - 1/x] = w(1 + x)/(x - w)$ for all $w \in \mathbb{C} \setminus [0, +\infty)$ and all $x \in (0, +\infty)$.

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In view of $z - \alpha \in \mathbb{C} \setminus [0, +\infty)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, we obtain thus

$$
\begin{aligned}
f(z) = g(z - \alpha) &= a + b(z - \alpha) + \int_{(0,+\infty)} \left[\frac{1}{x - (z - \alpha)} - \frac{1}{x} \right] \tau(\mathrm{d}x) \\
&= a + (z - \alpha)\left[b + \int_{(0,+\infty)} \frac{1+x}{x - (z - \alpha)} \chi(\mathrm{d}x) \right] \\
&= -d + (z - \alpha)\left[e + \int_{(\alpha,+\infty)} \frac{1 + t - \alpha}{t - z} \rho(\mathrm{d}t) \right]
\end{aligned}
$$

for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, where $d := -a$, $e := b$, and ρ is the image measure of χ under the translation $T \colon (0, +\infty) \to (\alpha, +\infty)$ defined by $T(x) := x + \alpha$. In particular $d, e \in [0, +\infty)$ and $\rho \in \mathcal{M}^1_{\geq}((\alpha, +\infty))$. Hence, the triple (d, e, ρ) is unique by virtue of Lem. 6.8. $\square$

Theorem 6.10. *Let $\alpha \in \mathbb{R}$ and let $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$. Then:*

(a) *If $F \in \mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$, then there are unique non-negative Hermitian complex $q \times q$ matrices D and E and a unique non-negative Hermitian measure $\rho \in \mathcal{M}^q_{\geq}((\alpha, +\infty))$ such that*

$$
F(z) = -D + (z - \alpha)\left[E + \int_{(\alpha,+\infty)} \frac{1 + t - \alpha}{t - z} \rho(\mathrm{d}t) \right] \tag{6.7}
$$

for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Furthermore, the function $P \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ defined by $P(z) := (z - \alpha)^{-1}F(z)$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ with $D = \mu_P(\{\alpha\})$ and $(E, \rho) = (\gamma_P, \mathrm{Rstr}_{\mathfrak{B}_{(\alpha,+\infty)}} \mu_P)$.

(b) *If $D \in \mathbb{C}^{q \times q}_{\geq}$, $E \in \mathbb{C}^{q \times q}_{\geq}$, and $\rho \in \mathcal{M}^q_{\geq}((\alpha, +\infty))$ are such that F can be represented via (6.7) for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then F belongs to $\mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$.*

Proof. (a) We consider an arbitrary vector $u \in \mathbb{C}^q$. According to Rem. 6.6, then $f_u := u^* F u$ belongs to $\mathcal{S}^{[-1]}_{1;[\alpha,+\infty)}$. Hence, Lem. 6.9 yields the existence of a unique triple $(d_u, e_u, \rho_u) \in [0, +\infty) \times [0, +\infty) \times \mathcal{M}^1_{\geq}((\alpha, +\infty))$ such that

$$
f_u(z) = -d_u + (z - \alpha)\left[e_u + \int_{(\alpha,+\infty)} \frac{1 + t - \alpha}{t - z} \rho_u(\mathrm{d}t) \right] \tag{6.8}
$$

holds true for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. With the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q)$ of $\mathbb{C}^q$, let $d_{jk} := \frac{1}{4} \sum_{\ell=0}^3 (-\mathrm{i})^\ell d_{\mathbf{e}_j + \mathrm{i}^\ell \mathbf{e}_k}$, $e_{jk} := \frac{1}{4} \sum_{\ell=0}^3 (-\mathrm{i})^\ell e_{\mathbf{e}_j + \mathrm{i}^\ell \mathbf{e}_k}$ and $\rho_{jk} := \frac{1}{4} \sum_{\ell=0}^3 (-\mathrm{i})^\ell \rho_{\mathbf{e}_j + \mathrm{i}^\ell \mathbf{e}_k}$

for all $j, k \in \mathbb{Z}_{1,q}$. We have then

$$
\begin{aligned}
\mathbf{e}_j^*[F(z)]\mathbf{e}_k &= \frac{1}{4}\sum_{\ell=0}^{3}(-\mathrm{i})^\ell f_{\mathbf{e}_j+\mathrm{i}^\ell\mathbf{e}_k}(z) \\
&= \frac{1}{4}\sum_{\ell=0}^{3}(-\mathrm{i})^\ell\left(-d_{\mathbf{e}_j+\mathrm{i}^\ell\mathbf{e}_k} + (z-\alpha)\left[e_{\mathbf{e}_j+\mathrm{i}^\ell\mathbf{e}_k} + \int_{(\alpha,+\infty)}\frac{1+t-\alpha}{t-z}\rho_{\mathbf{e}_j+\mathrm{i}^\ell\mathbf{e}_k}(\mathrm{d}t)\right]\right) \\
&= -d_{jk} + (z-\alpha)\left[e_{jk} + \int_{(\alpha,+\infty)}\frac{1+t-\alpha}{t-z}\rho_{jk}(\mathrm{d}t)\right]
\end{aligned}
$$

for all $j, k \in \mathbb{Z}_{1,q}$ and all $z \in \mathbb{C}\setminus[\alpha,+\infty)$. Hence, (6.7) follows for all $z \in \mathbb{C}\setminus[\alpha,+\infty)$ with $D := [d_{jk}]_{j,k=1}^q$, $E := [e_{jk}]_{j,k=1}^q$, and $\rho := [\rho_{jk}]_{j,k=1}^q$. For all $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, we have $f_{\zeta u} = f_u$ and thus $d_{\zeta u} = d_u$, $e_{\zeta u} = e_u$, and $\rho_{\zeta u} = \rho_u$ by virtue of the uniqueness of the triple (d_u, e_u, ρ_u). A straightforward calculation yields for all $j, k \in \mathbb{Z}_{1,q}$ then $d_{kj} = \overline{d_{jk}}$, $e_{kj} = \overline{e_{jk}}$, and $\rho_{kj}(B) = \overline{\rho_{jk}(B)}$ for all $B \in \mathfrak{B}_{(\alpha,+\infty)}$. Thus, the matrices D and E are Hermitian and ρ is an σ-additive mapping defined on $\mathfrak{B}_{(\alpha,+\infty)}$ with values in $\mathbb{C}_{\mathrm{H}}^{q\times q}$. From (6.7) we obtain $f_u(z) = -u^*Du + (z-\alpha)(u^*Eu + \int_{(\alpha,+\infty)}[(1+t-\alpha)/(t-z)](u^*\rho u)(\mathrm{d}t))$ for all $z \in \mathbb{C}\setminus[\alpha,+\infty)$, where u^*Du and u^*Eu belong to $\mathbb{R}$ and $u^*\rho u$ is a finite signed measure on $((\alpha,+\infty), \mathfrak{B}_{(\alpha,+\infty)})$. In view of (6.8), Lem. 6.8 yields $u^*Du = d_u$, $u^*Eu = e_u$, and $u^*\rho u = \rho_u$. In particular, u^*Du and u^*Eu belong to $[0,+\infty)$ and $u^*\rho u \in \mathcal{M}_{\geq}^1((\alpha,+\infty))$. Since $u \in \mathbb{C}^q$ was arbitrarily chosen, hence $D, E \in \mathbb{C}_{\geq}^{q\times q}$, and $\rho \in \mathcal{M}_{\geq}^q((\alpha,+\infty))$ follow.

Now let $D, E \in \mathbb{C}_{\geq}^{q\times q}$, and $\rho \in \mathcal{M}_{\geq}^q((\alpha,+\infty))$ be such that (6.7) holds true for all $z \in \mathbb{C}\setminus[\alpha,+\infty)$. Denote by δ_α the Dirac measure on $([\alpha,+\infty), \mathfrak{B}_{[\alpha,+\infty)})$ with unit mass at α. Then P admits the representation $P(z) = E + \int_{[\alpha,+\infty)}(1+t-\alpha)/(t-z)\theta(\mathrm{d}t)$ for all $z \in \mathbb{C}\setminus[\alpha,+\infty)$, where $\theta\colon \mathfrak{B}_{[\alpha,+\infty)} \to \mathbb{C}_{\geq}^{q\times q}$ defined by $\theta(B) := \rho(B\cap(\alpha,+\infty)) + [\delta_\alpha(B)]D$ belongs to $\mathcal{M}_{\geq}^q([\alpha,+\infty))$. Hence, Thm. 3.6(b) and Rem. 3.7 yield $P \in \mathcal{S}_{q;[\alpha,+\infty)}$ with $\gamma_P = E$ and $\mu_P = \theta$. In particular, $\mu_P(\{\alpha\}) = D$ and $\mathrm{Rstr}_{\mathfrak{B}_{(\alpha,+\infty)}}\mu_P = \rho$. Hence, the triple (D, E, ρ) is unique.

(b) Let $D, E \in \mathbb{C}_{\geq}^{q\times q}$ and $\rho \in \mathcal{M}_{\geq}^q((\alpha,+\infty))$ be such that (6.7) holds true for all $z \in \mathbb{C}\setminus[\alpha,+\infty)$. As explained above, then P belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$. Since $F(z) = (z-\alpha)P(z)$ for all $z \in \mathbb{C}\setminus[\alpha,+\infty)$, we hence conclude with Prop. 4.3 that F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$. $\square$

In the following, if $\alpha \in \mathbb{R}$ and $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ are given, then we will write (D_F, E_F, ρ_F) for the unique triple (D, E, ρ) from $\mathbb{C}_{\geq}^{q\times q} \times \mathbb{C}_{\geq}^{q\times q} \times \mathcal{M}_{\geq}^q((\alpha,+\infty))$ which fulfills (6.7) for all $z \in \mathbb{C}\setminus[\alpha,+\infty)$.

Corollary 6.11. *Let* $\alpha \in \mathbb{R}$. *Then* $F\colon \mathbb{C}\setminus[\alpha,+\infty) \to \mathbb{C}^{q\times q}$ *belongs to* $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ *if and only if* $P\colon \mathbb{C}\setminus[\alpha,+\infty) \to \mathbb{C}^{q\times q}$ *defined by* $P(z) = (z-\alpha)^{-1}F(z)$ *belongs to* $\mathcal{S}_{q;[\alpha,+\infty)}$.

Proof. If $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$, then $P \in \mathcal{S}_{q;[\alpha,+\infty)}$, by virtue of Thm. 6.10(a).

Conversely, now suppose $P \in \mathcal{S}_{q;[\alpha,+\infty)}$. According to Thm. 3.6 and Rem. 3.7, then

$$F(z) = (z - \alpha)P(z) = (z - \alpha)\left[\gamma_P + \int_{[\alpha,+\infty)} \frac{1 + t - \alpha}{t - z}\mu_P(\mathrm{d}t)\right]$$

$$= -D + (z - \alpha)\left[E + \int_{(\alpha,+\infty)} \frac{1 + t - \alpha}{t - z}\rho(\mathrm{d}t)\right]$$

for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$, where the matrices $D := \mu_P(\{\alpha\})$ and $E := \gamma_P$ are non-negative Hermitian and $\rho := \mathrm{Rstr}_{\mathfrak{B}_{(\alpha,+\infty)}} \mu_P$ belongs to $\mathcal{M}^q_{\geq}((\alpha, +\infty))$. Hence, Thm. 6.10(b) yields $F \in \mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$. $\qquad\square$

Corollary 6.12. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$. For all $x_1, x_2 \in (-\infty, \alpha)$ with $x_1 \leq x_2$, then $F(x_1) \leq F(x_2) \leq O_{q \times q}$.*

Proof. Using Thm. 6.10, we obtain

$$F(x_2) - F(x_1) = (x_2 - x_1)\left[E_F + \int_{[\alpha,+\infty)} \frac{(1 + t - \alpha)(t - \alpha)}{(t - x_2)(t - x_1)}\rho_F(\mathrm{d}t)\right]$$

for all $x_1, x_2 \in (-\infty, \alpha)$ with $x_1 \leq x_2$, by direct calculation. Since $E_F \in \mathbb{C}^{q \times q}_{\geq}$ and $-F(x) \in \mathbb{C}^{q \times q}_{\geq}$ for all $x \in (-\infty, \alpha)$, thus the proof is complete. $\qquad\square$

Now we consider again the situation of Example 6.7.

Example 6.13. Let $\alpha \in \mathbb{R}$ and let $D, E \in \mathbb{C}^{q \times q}_{\geq}$. Then $F: \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -D + (z - \alpha)E$ belongs to $\mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$, where $D_F = D$, $E_F = E$, and ρ_F is the constant measure with value $O_{q \times q}$.

Proposition 6.14. *Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$. Then:*

(a) *If $z \in \mathbb{C} \setminus [\alpha, +\infty)$, then $\overline{z} \in \mathbb{C} \setminus [\alpha, +\infty)$ and $[F(z)]^* = F(\overline{z})$.*

(b) *For all $z \in \mathbb{C} \setminus [\alpha, +\infty)$,*

$$\mathcal{N}(F(z)) = \mathcal{N}(D_F) \cap \mathcal{N}(E_F) \cap \mathcal{N}(\rho_F((\alpha, +\infty))), \qquad (6.9)$$
$$\mathcal{R}(F(z)) = \mathcal{R}(D_F) + \mathcal{R}(E_F) + \mathcal{R}(\rho_F((\alpha, +\infty))), \qquad (6.10)$$

and, in particular, $\mathcal{N}([F(z)]^*) = \mathcal{N}(F(z))$ and $\mathcal{R}([F(z)]^*) = \mathcal{R}(F(z))$.

(c) *Let $r \in \mathbb{N}_0$. Then the following statements are equivalent:*

   (i) $\mathrm{rank}\, F(z) = r$ *for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.*

   (ii) *There is some $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$ such that $\mathrm{rank}\, F(z_0) = r$.*

   (iii) $\dim[\mathcal{R}(D_F) + \mathcal{R}(E_F) + \mathcal{R}(\rho_F((\alpha, +\infty)))] = r$.

Proof. (a) This can be seen from Thm. 6.10(a).

(b) According to Thm. 6.10(a), the function $P\colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q\times q}$ defined by $P(z) := (z - \alpha)^{-1}F(z)$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$ with $D_F = \mu_P(\{\alpha\})$ and $(E_F, \rho_F) = (\gamma_P, \mathrm{Rstr}_{\mathfrak{B}_{(\alpha,+\infty)}} \mu_P)$. In particular, $\mu_P([\alpha, +\infty)) = D_F + \rho_F((\alpha, +\infty))$. Since the two matrices on the right-hand side of the last equation are both non-negative Hermitian, we get $\mathcal{N}(\mu_P([\alpha, +\infty))) = \mathcal{N}(D_F) \cap \mathcal{N}(\rho_F((\alpha, +\infty)))$. Now let $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Applying Prop. 3.15(b) to P, we get $\mathcal{N}(P(z)) = \mathcal{N}(\gamma_P) \cap \mathcal{N}(\mu_P([\alpha, +\infty)))$. In view of $\mathcal{N}(F(z)) = \mathcal{N}(P(z))$, then (6.9) follows. Thus, (6.9) is proved for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. From (a) and (6.9) we get $\mathcal{N}([F(z)]^*) = \mathcal{N}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. Taking additionally into account that the matrices D_F, E_F, and $\rho_F((\alpha, +\infty))$ are non-negative Hermitian, we obtain (6.10) from (6.9) in the same way as in the proof of Prop. 3.15(b). Using (a) and (6.10), we get $\mathcal{R}([F(z)]^*) = \mathcal{R}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

(c) This is a consequence of (6.10). $\qquad\square$

Corollary 6.15. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$, and let $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$. Then $F(z_0) = O_{q\times q}$ if and only if $F(z) = O_{q\times q}$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$.*

Proof. This is an immediate consequence of Prop. 6.14(c). $\qquad\square$

Corollary 6.16. *Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$, and let $\lambda \in \mathbb{R}$ be such that the matrix $D_F + \lambda I_q$ is non-negative Hermitian. Then*

$$\mathcal{R}(F(z) - \lambda I_q) = \mathcal{R}(F(w) - \lambda I_q) \quad and \quad \mathcal{N}(F(z) - \lambda I_q) = \mathcal{N}(F(w) - \lambda I_q) \quad (6.11)$$

for every choice of z and w in $\mathbb{C} \setminus [\alpha, +\infty)$. In particular, if $\lambda \geq 0$, then λ is an eigenvalue of the matrix $F(z_0)$ for some $z_0 \in \mathbb{C} \setminus [\alpha, +\infty)$ if and only if λ is an eigenvalue of the matrix $F(z)$ for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$. In this case, the eigenspaces $\mathcal{N}(F(z) - \lambda I_q)$ are independent of $z \in \mathbb{C} \setminus [\alpha, +\infty)$.

Proof. In view of Thm. 6.10, we conclude that the function $G\colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q\times q}$ defined by $G(z) := F(z) - \lambda I_q$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$. The application of Prop. 6.14(b) to the function G yields then (6.11). Since the matrix D_F is non-negative Hermitian, we have $D_F + \lambda I_q \in \mathbb{C}_{\geq}^{q\times q}$ if $\lambda \geq 0$. Thus, the remaining assertions are an immediate consequence of (6.11). $\qquad\square$

Lemma 6.17. *Let $\alpha \in \mathbb{R}$ and $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$. Then $F^\dagger$ is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$.*

Proof. In view of (6.9) and (6.10), we obtain for all $z \in \mathbb{C} \setminus [\alpha, +\infty)$ the identities $\mathcal{N}(F(z)) = \mathcal{N}(F(\mathrm{i}))$ and $\mathcal{R}(F(z)) = \mathcal{R}(F(\mathrm{i}))$. Thus, the application of [15, Prop. 8.4] completes the proof. $\qquad\square$

The following proposition is a generalization of a result due to Kats and Krein [17, Lem. D1.5.2], who considered the case $q = 1$ and $\alpha = 0$.

Theorem 6.18. *Let $\alpha \in \mathbb{R}$ and let $F\colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Then F belongs to $\mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$ if and only if $G := -F^\dagger$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$.*

Proof. First suppose $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$. Then G is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$ by virtue of Lem. 6.17. In view of $\mathrm{Rstr}_{\Pi_+} F \in \mathcal{R}_q(\Pi_+)$, we conclude from [11, Prop. 3.8] that $\mathrm{Rstr}_{\Pi_+} G$ belongs to $\mathcal{R}_q(\Pi_+)$ as well. In particular, $\mathrm{Im}\, G(w) \in \mathbb{C}_{\geq}^{q \times q}$ for all $w \in \Pi_+$. Because of $-F(x) \in \mathbb{C}_{\geq}^{q \times q}$ for each $x \in (-\infty, \alpha)$, we have $G(x) = [-F(x)]^\dagger \in \mathbb{C}_{\geq}^{q \times q}$ for all $x \in (-\infty, \alpha)$ (see, e.g. [5, Lem. 1.1.5]). Hence, G belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$.

Now suppose $G \in \mathcal{S}_{q;[\alpha,+\infty)}$. Then $F = -G^\dagger$. Hence, F is holomorphic in $\mathbb{C} \setminus [\alpha, +\infty)$ by virtue of Lem. 6.1. Since Prop. 4.3 yields $\mathrm{Rstr}_{\Pi_+} G \in \mathcal{R}_q(\Pi_+)$, we conclude from [11, Prop. 3.8] that $\mathrm{Rstr}_{\Pi_+} F$ belongs to $\mathcal{R}_q(\Pi_+)$ as well. Because of $G(x) \in \mathbb{C}_{\geq}^{q \times q}$ for all $x \in (-\infty, \alpha)$, we have $-F(x) = [G(x)]^\dagger \in \mathbb{C}_{\geq}^{q \times q}$ for all $x \in (-\infty, \alpha)$ (see, e.g. [5, Lem. 1.1.5]). Hence, $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$. $\square$

7. Integral representations for the class $\mathcal{S}_{q;(-\infty,\beta]}$

The main goal of this section is to derive some integral representations for $(-\infty, \beta]$-Stieltjes functions of order q. Our strategy is based on using the corresponding results for the class $\mathcal{S}_{q;[-\beta,+\infty)}$. The following observation provides the key to realize our aims.

Remark 7.1. Let $\alpha, \beta \in \mathbb{R}$ and let $T \colon \mathbb{C} \to \mathbb{C}$ be defined by $T(z) := \alpha + \beta - \overline{z}$. Then $T(\mathbb{C} \setminus [\alpha, +\infty)) = \mathbb{C} \setminus (-\infty, \beta]$, $T(\mathbb{C} \setminus (-\infty, \beta]) = \mathbb{C} \setminus [\alpha, +\infty)$, $T(\Pi_+) = \Pi_+$, $T((-\infty, \alpha)) = (\beta, +\infty)$ and $T((\beta, +\infty)) = (-\infty, \alpha)$. Consequently, in view of Prop. 4.4, one can easily check that, for each $F \in \mathcal{S}_{q;[\alpha,+\infty)}$, the function $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $G(z) := -[F(\alpha + \beta - \overline{z})]^*$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ and that, conversely, for any function $G \in \mathcal{S}_{q;(-\infty,\beta]}$, the function $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(\alpha + \beta - \overline{z})]^*$ belongs to $\mathcal{S}_{q;[\alpha,+\infty)}$.

Proposition 7.2. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}$. Then the Nevanlinna parametrization (A, B, ν) of $\mathrm{Rstr}_{\Pi_+} G$ fulfills $\nu((\beta, +\infty)) = O_{q \times q}$, $B = O_{q \times q}$, and $\nu \in \mathcal{M}_{\geq,1}^q(\mathbb{R})$. In particular, for each $z \in \mathbb{C} \setminus (-\infty, \beta]$, then $G(z) = A + \int_{(-\infty,\beta]}(1 + tz)/(t - z)\nu(\mathrm{d}t)$.*

Proof. According to Rem. 7.1, the function $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$. From Rem. 2.15 we obtain then that the Nevanlinna parametrization of $\mathrm{Rstr}_{\Pi_+} F$ is given by $(-A, B, \theta)$, where θ is the image measure of ν under the reflection $t \mapsto -t$ on $\mathbb{R}$. Now Prop. 2.16 yields $\theta((-\infty, -\beta)) = O_{q \times q}$, $B = O_{q \times q}$, $\theta \in \mathcal{M}_{\geq,1}^q(\mathbb{R})$, and $F(z) = -A + \int_{[-\beta,+\infty)}(1 + tz)/(t - z)\theta(\mathrm{d}t)$ for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$. Hence, $\nu((\beta, +\infty)) = O_{q \times q}$, $\nu \in \mathcal{M}_{\geq,1}^q(\mathbb{R})$, and

$$G(z) = -[F(-\overline{z})]^* = A + \int_{[-\beta,+\infty)} \frac{1 - tz}{-t - z}\theta(\mathrm{d}t) = A + \int_{(-\infty,\beta]} \frac{1 + tz}{t - z}\nu(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. $\square$

Theorem 7.3. *Let $\beta \in \mathbb{R}$ and let $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$. Then:*

7. Integral representations for the class $\mathcal{S}_{q;(-\infty,\beta]}$

(a) *Suppose $G \in \mathcal{S}_{q;(-\infty,\beta]}$. Denote by (A, B, ν) the Nevanlinna parametrization of $\mathrm{Rstr}_{\Pi_+} G$ and let $\tilde{\nu} := \mathrm{Rstr}_{\mathfrak{B}_{(-\infty,\beta]}} \nu$. Then $\tilde{\nu} \in \mathcal{M}_{\geq,1}^q((-\infty,\beta])$ and there is a unique pair (C, η) from $\mathbb{C}_{\geq}^{q \times q} \times \mathcal{M}_{\geq,1}^q((-\infty,\beta])$ such that*

$$G(z) = -C + \int_{(-\infty,\beta]} \frac{1+t^2}{t-z} \eta(\mathrm{d}t) \tag{7.1}$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, namely $C = \int_{(-\infty,\beta]} t\tilde{\nu}(\mathrm{d}t) - A$ and $\eta = \tilde{\nu}$. Furthermore, $C = -\lim_{r \to +\infty} G(\beta + re^{i\phi})$ for all $\phi \in (-\pi/2, \pi/2)$.

(b) *Let $C \in \mathbb{C}_{\geq}^{q \times q}$ and let $\eta \in \mathcal{M}_{\geq,1}^q((-\infty,\beta])$ be such that (7.1) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Then G belongs to $\mathcal{S}_{q;(-\infty,\beta]}$.*

Proof. (a) According to Rem. 7.1, the function $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$. From Rem. 2.15 we obtain then that the Nevanlinna parametrization of $\mathrm{Rstr}_{\Pi_+} F$ is given by $(-A, B, \theta)$, where θ is the image measure of ν under the reflection $t \mapsto -t$ on $\mathbb{R}$. Now Thm. 3.1(a) yields that $\tilde{\theta} := \mathrm{Rstr}_{\mathfrak{B}_{[-\beta,+\infty)}} \theta$ belongs to $\mathcal{M}_{\geq,1}^q([-\beta,+\infty))$ and that there is a unique pair $(D, \tau) \in \mathbb{C}_{\geq}^{q \times q} \times \mathcal{M}_{\geq,1}^q([-\beta,+\infty))$ such that $F(z) = D + \int_{[-\beta,+\infty)}(1+t^2)/(t-z)\tau(\mathrm{d}t)$ for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, namely $D = -A - \int_{[-\beta,+\infty)} t\tilde{\theta}(\mathrm{d}t)$ and $\tau = \tilde{\theta}$. Since $\tilde{\theta}$ is the image measure of $\tilde{\nu}$ under the transformation $T \colon (-\infty, \beta] \to [-\beta, +\infty)$ defined by $T(t) := -t$, we can conclude $\tilde{\nu} \in \mathcal{M}_{\geq,1}^q((-\infty,\beta])$ and

$$G(z) = -[F(-\overline{z})]^* = -D^* - \left[\int_{[-\beta,+\infty)} \frac{1+t^2}{t+\overline{z}} \tau(\mathrm{d}t)\right]^* = -D - \int_{[-\beta,+\infty)} \frac{1+t^2}{t+z} \tau(\mathrm{d}t)$$

$$= -\left[\int_{[-\beta,+\infty)} (-t)\tilde{\theta}(\mathrm{d}t) - A\right] + \int_{[-\beta,+\infty)} \frac{1+t^2}{-t-z} \tilde{\theta}(\mathrm{d}t) = -C + \int_{(-\infty,\beta]} \frac{1+t^2}{t-z} \eta(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, where $C := \int_{(-\infty,\beta]} t\tilde{\nu}(\mathrm{d}t) - A$ and $\eta := \tilde{\nu}$. From the above computation we see $C = D$ and hence $C \in \mathbb{C}_{\geq}^{q \times q}$ follows. Taking additionally into account Thm. 3.1(a), for all $\phi \in (-\pi/2, \pi/2)$, we get

$$C = D^* = \left[\lim_{r \to +\infty} F(-\beta + re^{i(\pi-\phi)})\right]^* = \lim_{r \to +\infty} \left[F(-\beta - re^{-i\phi})\right]^* = -\lim_{r \to +\infty} G(\beta + re^{i\phi}).$$

Now let $C \in \mathbb{C}_{\geq}^{q \times q}$ and $\eta \in \mathcal{M}_{\geq,1}^q((-\infty,\beta])$ be such that (7.1) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Then $\chi \colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}_{\geq}^{q \times q}$ defined by $\chi(M) := \eta(M \cap (-\infty, \beta])$ belongs to $\mathcal{M}_{\geq,1}^q(\mathbb{R})$ and the matrix $-C + \int_{\mathbb{R}} t\chi(\mathrm{d}t)$ is Hermitian. Using (3.2), we conclude from (7.1) then that the integral $\int_{\mathbb{R}}(1+tz)/(t-z)\chi(\mathrm{d}t)$ exists and that $G(z) = -C + \int_{\mathbb{R}} t\chi(\mathrm{d}t) + z \cdot O_{q \times q} + \int_{\mathbb{R}}(1+tz)/(t-z)\chi(\mathrm{d}t)$ for all $z \in \Pi_+$. Thm. 2.13(a) yields then $-C + \int_{\mathbb{R}} t\chi(\mathrm{d}t) = A$ and $\chi = \nu$. Hence $\eta = \tilde{\nu}$ and $C = \int_{(-\infty,\beta]} t\tilde{\nu}(\mathrm{d}t) - A$.

(b) Let $C \in \mathbb{C}_{\geq}^{q \times q}$ and $\eta \in \mathcal{M}_{\geq,1}^q((-\infty,\beta])$ be such that (7.1) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Using a result on holomorphic dependence of an integral on a complex

parameter (see, e.g. [9, Ch. IV, §5, Satz 5.8]), we conclude then that G is a matrix-valued function which is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$. Furthermore,

$$\operatorname{Im} G(z) = \int_{(-\infty,\beta]} \operatorname{Im}\left(\frac{1+t^2}{t-z}\right)\eta(\mathrm{d}t) = \int_{(-\infty,\beta]} \frac{(1+t^2)\operatorname{Im} z}{|t-z|^2}\eta(\mathrm{d}t) \in \mathbb{C}_{\geq}^{q \times q}$$

for all $z \in \Pi_+$ and

$$-G(x) = C + \int_{(-\infty,\beta]} \frac{1+t^2}{x-t}\eta(\mathrm{d}t) \in \mathbb{C}_{\geq}^{q \times q}$$

for all $x \in (\beta, +\infty)$. Thus, G belongs to $\mathcal{S}_{q;(-\infty,\beta]}$. $\qquad\square$

In the following, if $\beta \in \mathbb{R}$ and $G \in \mathcal{S}_{q;(-\infty,\beta]}$ are given, then we will write (C_G, η_G) for the unique pair (C, η) from $\mathbb{C}_{\geq}^{q \times q} \times \mathcal{M}_{\geq,1}^q((-\infty,\beta])$ which fulfills (7.1) for all $z \in \mathbb{C} \setminus (-\infty, \beta]$.

Remark 7.4. Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}$. For all $x_1, x_2 \in (\beta, +\infty)$ with $x_1 \leq x_2$, then $G(x_1) \leq G(x_2) \leq O_{q \times q}$, by virtue of Thm. 7.3(a).

Remark 7.5. (a) Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. In view of Thm. 3.1, we have then

$$-[F(-\overline{z})]^* = -C_F + \int_{[\alpha,+\infty)} \frac{1+t^2}{-t-z}\eta_F(\mathrm{d}t) = -C_F + \int_{(-\infty,-\alpha]} \frac{1+t^2}{t-z}\hat{\theta}(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus (-\infty, -\alpha]$, where $\hat{\theta}$ is the image measure of η_F under the transformation $R: [\alpha, +\infty) \to (-\infty, -\alpha]$ defined by $R(t) := -t$. Because of $C_F \in \mathbb{C}_{\geq}^{q \times q}$ and $\hat{\theta} \in \mathcal{M}_{\geq,1}^q((-\infty, -\alpha])$, Thm. 7.3 yields then that $G: \mathbb{C} \setminus (-\infty, -\alpha] \to \mathbb{C}^{q \times q}$ defined by $G(z) := -[F(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;(-\infty,-\alpha]}$ and that $(C_G, \eta_G) = (C_F, \hat{\theta})$.

(b) Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}$. In view of Thm. 7.3 and Prop. A.5, we have then

$$-[G(-\overline{z})]^* = C_G + \int_{(-\infty,\beta]} \frac{1+t^2}{-t-z}\eta_G(\mathrm{d}t) = C_G + \int_{[-\beta,+\infty)} \frac{1+t^2}{t-z}\tilde{\theta}(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, where $\tilde{\theta}$ is the image measure of η_G under the transformation $T: (-\infty, \beta] \to [-\beta, +\infty)$ defined by $T(t) := -t$. Because of $C_G \in \mathbb{C}_{\geq}^{q \times q}$ and $\tilde{\theta} \in \mathcal{M}_{\geq,1}^q([-\beta, +\infty))$, Thm. 3.1 yields then that $F: \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ and that $(C_F, \eta_F) = (C_G, \tilde{\theta})$.

Now we get an integral representation for functions which belong to the class $\mathcal{S}_{q;(-\infty,\beta]}$.

Theorem 7.6. *Let $\beta \in \mathbb{R}$ and let $G: \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$. Then:*

(a) *If $G \in \mathcal{S}_{q;(-\infty,\beta]}$, then there are a unique matrix $\gamma \in \mathbb{C}_{\geq}^{q \times q}$ and a unique non-negative Hermitian measure $\mu \in \mathcal{M}_{\geq}^q((-\infty,\beta])$ such that*

$$G(z) = -\gamma + \int_{(-\infty,\beta]} \frac{1+\beta-t}{t-z}\mu(\mathrm{d}t). \qquad (7.2)$$

holds for each $z \in \mathbb{C} \setminus (-\infty, \beta]$. Furthermore, $\gamma = C_G$ and $\gamma = -\lim_{y \to +\infty} G(\mathrm{i}y)$.

(b) *If there are a matrix $\gamma \in \mathbb{C}_\succeq^{q \times q}$ and a non-negative Hermitian measure $\mu \in \mathcal{M}_\succeq^q([\alpha, +\infty))$ such that G can be represented via (7.2) for each $z \in \mathbb{C} \setminus (-\infty, \beta]$, then G belongs to the class $\mathcal{S}_{q;(-\infty,\beta]}$.*

Proof. (a) According to Rem. 7.5(b), the function $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ and $C_F = C_G$. Thm. 3.6(a) yields then that there is a unique pair (δ, ρ) from $\mathbb{C}_\succeq^{q \times q} \times \mathcal{M}_\succeq^q([-\beta, +\infty))$ such that

$$F(z) = \delta + \int_{[-\beta,+\infty)} \frac{1 + t + \beta}{t - z} \rho(\mathrm{d}t) \tag{7.3}$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$ and that $\delta = C_F$. Applying Prop. A.5, we now infer

$$G(z) = -[F(-\overline{z})]^* = -C_F - \int_{[-\beta,+\infty)} \frac{1 + t + \beta}{t + z} \rho(\mathrm{d}t)$$

$$= -C_G + \int_{[-\beta,+\infty)} \frac{1 + \beta - (-t)}{-t - z} \rho(\mathrm{d}t) = -\gamma + \int_{(-\infty,\beta]} \frac{1 + \beta - t}{t - z} \mu(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, where $\gamma := C_G$ and μ is the image measure of ρ under the transformation $R \colon [-\beta, +\infty) \to (-\infty, \beta]$ defined by $R(t) := -t$. Since Prop. 3.13 yields $\lim_{y \to +\infty} F(\mathrm{i}y) = \delta$, we conclude furthermore

$$\gamma = C_G = C_F = \delta = \delta^* = \left[\lim_{y \to +\infty} F(\mathrm{i}y) \right]^* = -\lim_{y \to +\infty} G(\mathrm{i}y).$$

Now let $\gamma \in \mathbb{C}_\succeq^{q \times q}$ and $\mu \in \mathcal{M}_\succeq^q((-\infty, \beta])$ be arbitrary such that (7.2) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Then using Prop. A.5 we get

$$F(z) = -[G(-\overline{z})]^* = \gamma - \int_{(-\infty,\beta]} \frac{1 + \beta - t}{t + z} \mu(\mathrm{d}t)$$

$$= \gamma + \int_{(-\infty,\beta]} \frac{1 - t + \beta}{-t - z} \mu(\mathrm{d}t) = \gamma + \int_{[-\beta,+\infty)} \frac{1 + t + \beta}{t - z} \tilde{\theta}(\mathrm{d}t) \tag{7.4}$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, where $\tilde{\theta}$ is the image measure of μ under the transformation $T \colon (-\infty, \beta] \to [-\beta, +\infty)$ defined by $T(t) := -t$. Since we know from Thm. 3.6(a) that the pair $(\delta, \rho) \in \mathbb{C}_\succeq^{q \times q} \times \mathcal{M}_\succeq^q([-\beta, +\infty))$ with (7.3) for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$ is unique, comparing with (7.4), we conclude $\gamma = \delta$ and $\tilde{\theta} = \rho$. Hence, $\gamma = C_F = C_G$ and μ is the image measure of ρ under the transformation R.

(b) Let $\gamma \in \mathbb{C}_\succeq^{q \times q}$ and $\mu \in \mathcal{M}_\succeq^q((-\infty, \beta])$ be such that (7.2) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Then $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ fulfills (7.4) for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, where $\tilde{\theta}$ is the image measure of μ under the transformation T. Thm. 3.6(b) yields then $F \in \mathcal{S}_{q;[-\beta,+\infty)}$. In view of $G(z) = -[F(-\overline{z})]^*$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, hence G belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ by virtue of Rem. 7.1. $\qquad\square$

In the following, if $\beta \in \mathbb{R}$ and $G \in \mathcal{S}_{q;(-\infty,\beta]}$ are given, then we will write (γ_G, μ_G) for the unique pair (γ, μ) from $\mathbb{C}_\succeq^{q \times q} \times \mathcal{M}_\succeq^q((-\infty, \beta])$ which fulfills (7.2) for all $z \in \mathbb{C} \setminus (-\infty, \beta]$.

Remark 7.7. (a) Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}$. In view of Thm. 3.6 and Prop. A.5, we have then

$$-[F(-\overline{z})]^* = -\gamma_F + \int_{[\alpha,+\infty)} \frac{1-(-t)-\alpha}{-t-z}\mu_F(\mathrm{d}t) = -\gamma_F + \int_{(-\infty,-\alpha]} \frac{1-\alpha-t}{t-z}\hat{\theta}(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus (-\infty,-\alpha]$, where $\hat{\theta}$ is the image measure of μ_F under the transformation $R \colon [\alpha,+\infty) \to (-\infty,-\alpha]$ defined by $R(t) := -t$. Because of $\gamma_F \in \mathbb{C}_{\geq}^{q\times q}$, Thm. 7.6 yields then that $G \colon \mathbb{C} \setminus (-\infty,-\alpha] \to \mathbb{C}^{q\times q}$ defined by $G(z) := -[F(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;(-\infty,-\alpha]}$ and that $(\gamma_G,\mu_G) = (\gamma_F,\hat{\theta})$.

(b) Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}$. In view of Thm. 7.6, we have then

$$-[G(-\overline{z})]^* = \gamma_G + \int_{(-\infty,\beta]} \frac{1+\beta-t}{-t-z}\mu_G(\mathrm{d}t) = \gamma_G + \int_{[-\beta,+\infty)} \frac{1+t+\beta}{t-z}\tilde{\theta}(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus [-\beta,+\infty)$, where $\tilde{\theta}$ is the image measure of μ_G under the transformation $T \colon (-\infty,\beta] \to [-\beta,+\infty)$ defined by $T(t) := -t$. Because of $\gamma_G \in \mathbb{C}_{\geq}^{q\times q}$, Thm. 3.6 yields then that $F \colon \mathbb{C} \setminus [-\beta,+\infty) \to \mathbb{C}^{q\times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ and that $(\gamma_F,\mu_F) = (\gamma_G,\tilde{\theta})$.

Proposition 7.8. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}$. Then:*

(a) *Let $z \in \mathbb{C} \setminus (-\infty,\beta]$. Then $\overline{z} \in \mathbb{C} \setminus (-\infty,\beta]$ and $[G(z)]^* = G(\overline{z})$.*

(b) *For all $z \in \mathbb{C} \setminus (-\infty,\beta]$,*

$$\mathcal{R}(G(z)) = \mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G((-\infty,\beta])), \quad \mathcal{N}(G(z)) = \mathcal{N}(\gamma_G) \cap \mathcal{N}(\mu_G((-\infty,\beta])),$$

(7.5)

and, in particular, $\mathcal{R}([G(z)]^*) = \mathcal{R}(G(z))$ and $\mathcal{N}([G(z)]^*) = \mathcal{N}(G(z))$.

(c) *Let $r \in \mathbb{N}_0$. Then the following statements are equivalent:*

(i) $\operatorname{rank} G(z) = r$ *for all $z \in \mathbb{C} \setminus (-\infty,\beta]$.*
(ii) *There is some $z_0 \in \mathbb{C} \setminus (-\infty,\beta]$ such that $\operatorname{rank} G(z_0) = r$.*
(iii) $\dim[\mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G((-\infty,\beta]))] = r$.

Proof. (a) This can be concluded from the representation (7.2) in Thm. 7.6(a).

(b) According to Rem. 7.7(b), the function $F \colon \mathbb{C} \setminus [-\beta,+\infty) \to \mathbb{C}^{q\times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ and $(\gamma_F,\mu_F) = (\gamma_G,\tilde{\theta})$, where $\tilde{\theta}$ is the image measure of μ_G under the transformation $T \colon (-\infty,\beta] \to [-\beta,+\infty)$ defined by $T(t) := -t$. In particular, $\mu_F([-\beta,+\infty)) = \mu_G((-\infty,\beta])$. Prop. 3.15(b) yields $\mathcal{R}([F(w)]^*) = \mathcal{R}(F(w)) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([-\beta,+\infty)))$ and $\mathcal{N}([F(w)]^*) = \mathcal{N}(F(w)) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([-\beta,+\infty)))$ for all $w \in \mathbb{C} \setminus [-\beta,+\infty)$. We now infer

$$\mathcal{R}(G(z)) = \mathcal{R}([F(-\overline{z})]^*) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([-\beta,+\infty))) = \mathcal{R}(\gamma_G) + \mathcal{R}(\mu_G((-\infty,\beta]))$$

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and

$$\mathcal{N}(G(z)) = \mathcal{N}([F(-\overline{z})]^*) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([-\beta,+\infty))) = \mathcal{N}(\gamma_G) \cap \mathcal{N}(\mu_G((-\infty,\beta]))$$

for all $z \in \mathbb{C} \setminus (-\infty,\beta]$. From (a) and (7.5) we get $\mathcal{R}([G(z)]^*) = \mathcal{R}(G(z))$ and $\mathcal{N}([G(z)]^*) = \mathcal{N}(G(z))$.

(c) This is an immediate consequence of (b). $\qquad\square$

Corollary 7.9. *Let $\beta \in \mathbb{R}$, let $G \in \mathcal{S}_{q;(-\infty,\beta]}$, and let $z_0 \in \mathbb{C} \setminus (-\infty,\beta]$. Then $G(z_0) = O_{q\times q}$ if and only if G is identically $O_{q\times q}$.*

Proof. This is an immediate consequence of Prop. 7.8(c). $\qquad\square$

Corollary 7.10. *Let $\beta \in \mathbb{R}$, let $G \in \mathcal{S}_{q;(-\infty,\beta]}$, and let $\lambda \in \mathbb{R}$ be such that the matrix $\gamma_G + \lambda I_q$ is non-negative Hermitian. Then*

$$\mathcal{R}(G(z) - \lambda I_q) = \mathcal{R}(G(w) - \lambda I_q) \quad and \quad \mathcal{N}(G(z) - \lambda I_q) = \mathcal{N}(G(w) - \lambda I_q) \quad (7.6)$$

for all $z, w \in \mathbb{C} \setminus [\alpha,+\infty)$. In particular, if $\lambda \geq 0$, then λ is an eigenvalue of the matrix $G(z_0)$ for some $z_0 \in \mathbb{C} \setminus [\alpha,+\infty)$ if and only if λ is an eigenvalue of the matrix $G(z)$ for all $z \in \mathbb{C} \setminus [\alpha,+\infty)$. In this case, the eigenspaces $\mathcal{N}(G(z) - \lambda I_q)$ are independent of $z \in \mathbb{C} \setminus [\alpha,+\infty)$.

Proof. In view of Thm. 7.6, we conclude that the function $F \colon \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := G(z) - \lambda I_q$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}$. The application of Prop. 7.8(b) to the function F yields then (7.6). Since the matrix γ_G is non-negative Hermitian, we have $\gamma_G + \lambda I_q \in \mathbb{C}_{\geq}^{q \times q}$ if $\lambda \geq 0$. Thus, the remaining assertions are an immediate consequence of (7.6). $\qquad\square$

Now we apply the preceding results to the subclass $\mathcal{S}_{q;(-\infty,\beta]}^{\diamond}$ of $\mathcal{S}_{q;(-\infty,\beta]}$.

Remark 7.11. Let $\beta \in \mathbb{R}$. From Thm. 7.6(a) we see that the class $\mathcal{S}_{q;(-\infty,\beta]}^{\diamond}$ consists of all $G \in \mathcal{S}_{q;(-\infty,\beta]}$ with $\gamma_G = O_{q\times q}$.

Remark 7.12. Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}^{\diamond}$. In view of Rem. 7.11 and Prop. 7.8(b), then $\mathcal{N}(G(z)) = \mathcal{N}(\mu_G((-\infty,\beta]))$ and $\mathcal{R}(G(z)) = \mathcal{R}(\mu_G((-\infty,\beta]))$ for all $z \in \mathbb{C} \setminus (-\infty,\beta]$.

8. Characterizations of the class $\mathcal{S}_{q;(-\infty,\beta]}$

While we have discussed the class $\mathcal{S}_{q;[\alpha,+\infty)}$ in Section 4, we give here now the corresponding results for the class $\mathcal{S}_{q;(-\infty,\beta]}$.

Proposition 8.1. *Let $\beta \in \mathbb{R}$, let $G \colon \mathbb{C} \setminus (-\infty,\beta] \to \mathbb{C}^{q \times q}$ be a matrix-valued function, and let $G^{\square} \colon \mathbb{C} \setminus (-\infty,\beta] \to \mathbb{C}^{q \times q}$ be defined by $G^{\square}(z) := (\beta - z)G(z)$. Then G belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ if and only if the following two conditions hold true:*

(I) *G is holomorphic in $\mathbb{C} \setminus (-\infty,\beta]$.*

(II) The matrix-valued functions $\mathrm{Rstr}_{\Pi_+} G$ *and* $\mathrm{Rstr}_{\Pi_+} G^{\square}$ *both belong to* $\mathcal{R}_q(\Pi_+)$.

Proof. According to Rem. 7.1, the function G belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ if and only if the function $F\colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q\times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$. Furthermore, G is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$ if and only if F is holomorphic in $\mathbb{C} \setminus [-\beta, +\infty)$. By virtue of Rem. 2.15, the function $\mathrm{Rstr}_{\Pi_+} G$ belongs to $\mathcal{R}_q(\Pi_+)$ if and only if $\mathrm{Rstr}_{\Pi_+} F$ belongs to $\mathcal{R}_q(\Pi_+)$. Let $F^{\square}\colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q\times q}$ be defined by $F^{\square}(z) := (z + \beta)F(z)$. Then

$$-\left[G^{\square}(-\overline{z})\right]^* = -[(\beta + \overline{z})G(-\overline{z})]^* = (z + \beta)(-[G(-\overline{z})]^*) = F^{\square}(z)$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$. According to Rem. 2.15, hence $\mathrm{Rstr}_{\Pi_+} G^{\square}$ belong to $\mathcal{R}_q(\Pi_+)$ if and only if $\mathrm{Rstr}_{\Pi_+} F^{\square}$ belong to $\mathcal{R}_q(\Pi_+)$. The application of Prop. 4.3 completes the proof. $\square$

For each $\beta \in \mathbb{R}$, let $\mathbb{C}_{\beta,+} := \{z \in \mathbb{C}\colon \mathrm{Re}\, z > \beta\}$.

Proposition 8.2. *Let* $\beta \in \mathbb{R}$ *and let* $G\colon \mathbb{C}\setminus(-\infty,\beta] \to \mathbb{C}^{q\times q}$ *be a matrix-valued function. Then* G *belongs to* $\mathcal{S}_{q;(-\infty,\beta]}$ *if and only if the following four conditions are fulfilled:*

(I) G is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$.

(II) $\mathrm{Im}\, G(z)$ is non-negative Hermitian for all $z \in \Pi_+$.

(III) $-\mathrm{Im}\, G(z)$ is non-negative Hermitian for all $z \in \Pi_-$.

(IV) $-\mathrm{Re}\, G(z)$ is non-negative Hermitian for all $z \in \mathbb{C}_{\beta,+}$.

Proof. According to Rem. 7.1, the function G belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ if and only if the function $F\colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q\times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$. Furthermore, G is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$ if and only if F is holomorphic in $\mathbb{C} \setminus [-\beta, +\infty)$. For all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, we have $\mathrm{Re}\, F(z) = -\mathrm{Re}\, G(-\overline{z})$ and $\mathrm{Im}\, F(z) = \mathrm{Im}\, G(-\overline{z})$. Hence,

$$\mathrm{Im}\, F(\Pi_+) = \mathrm{Im}\, G(\Pi_+), \quad \mathrm{Im}\, F(\Pi_-) = \mathrm{Im}\, G(\Pi_-) \quad \text{and} \quad \mathrm{Re}\, F(\mathbb{C}_{-\beta,-}) = -\mathrm{Re}\, G(\mathbb{C}_{\beta,+}).$$

The application of Prop. 4.4 completes the proof. $\square$

9. The class $\mathcal{S}_{0,q;(-\infty,\beta]}$

In Section 5, we have studied the class $\mathcal{S}_{0,q;[\alpha,+\infty)}$. The aim of this section is to derive corresponding results for the dual class $\mathcal{S}_{0,q;(-\infty,\beta]}$. The following observation establishes the bridge to Section 5.

Remark 9.1.　(a) Let $\alpha \in \mathbb{R}$, let $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$, and let $G\colon \mathbb{C} \setminus (-\infty, -\alpha] \to \mathbb{C}^{q\times q}$ be defined by $G(z) := -[F(-\overline{z})]^*$. We have then $\|G(\mathrm{i}y)\|_{\mathrm{E}} = \|F(\mathrm{i}y)\|_{\mathrm{E}}$ for all $y \in [1, +\infty)$. Taking additionally into account Rem. 7.1, one can see that G belongs to $\mathcal{S}_{0,q;(-\infty,-\alpha]}$.

(b) Let $\beta \in \mathbb{R}$, let $G \in \mathcal{S}_{0,q;(-\infty,\beta]}$, and let $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ be defined by $F(z) := -[G(-\overline{z})]^*$. We have then $\|F(\mathrm{i}y)\|_{\mathrm{E}} = \|G(\mathrm{i}y)\|_{\mathrm{E}}$ for all $y \in [1, +\infty)$. Taking additionally into account Rem. 7.1, one can see that F belongs to $\mathcal{S}_{0,q;[-\beta,+\infty)}$.

Theorem 9.2. *Let $\beta \in \mathbb{R}$ and let $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$. Then:*

(a) *If $G \in \mathcal{S}_{0,q;(-\infty,\beta]}$, then there is a unique measure $\sigma \in \mathcal{M}_{\geq}^q((-\infty, \beta])$ such that*

$$G(z) = \int_{(-\infty,\beta]} \frac{1}{t - z} \sigma(\mathrm{d}t) \tag{9.1}$$

for each $z \in \mathbb{C} \setminus (-\infty, \beta]$. Furthermore, $\sigma((-\infty, \beta]) = -\mathrm{i} \lim_{y \to +\infty} y G(\mathrm{i}y)$.

(b) *If there is a measure $\sigma \in \mathcal{M}_{\geq}^q((-\infty, \beta])$ such that G can be represented via (9.1) for each $z \in \mathbb{C} \setminus (-\infty, \beta]$, then G belongs to the class $\mathcal{S}_{0,q;(-\infty,\beta]}$.*

Proof. (a) According to Rem. 9.1(b), the function $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{0,q;[-\beta,+\infty)}$. Thm. 5.1(a) yields then the existence of a unique non-negative Hermitian measure $\tau \in \mathcal{M}_{\geq}^q([-\beta, +\infty))$ such that

$$F(z) = \int_{[-\beta,+\infty)} \frac{1}{t - z} \tau(\mathrm{d}t) \tag{9.2}$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$. Using Prop. A.5, we now infer

$$G(z) = -[F(-\overline{z})]^* = \int_{[-\beta,+\infty)} \frac{1}{-t - z} \tau(\mathrm{d}t) = \int_{(-\infty,\beta]} \frac{1}{t - z} \sigma(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, where σ is the image measure of τ under the transformation $R \colon [-\beta, +\infty) \to (-\infty, \beta]$ defined by $R(t) := -t$. Since Prop. 5.3 yields $\tau([-\beta, +\infty)) = -\mathrm{i} \lim_{y \to +\infty} y F(\mathrm{i}y)$, we conclude furthermore

$$\sigma((-\infty, \beta]) = [\sigma((-\infty, \beta])]^* = [\tau([-\beta, +\infty))]^* = \mathrm{i} \lim_{y \to +\infty} y[F(\mathrm{i}y)]^* = -\mathrm{i} \lim_{y \to +\infty} y G(\mathrm{i}y).$$

Now let $\sigma \in \mathcal{M}_{\geq}^q((-\infty, \beta])$ be such that (9.1) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Using Prop. A.5, we get then

$$F(z) = -[G(-\overline{z})]^* = \int_{(-\infty,\beta]} \frac{1}{-t - z} \sigma(\mathrm{d}t) = \int_{[-\beta,+\infty)} \frac{1}{t - z} \tilde{\theta}(\mathrm{d}t) \tag{9.3}$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, where $\tilde{\theta}$ is the image measure of σ under the transformation $T \colon (-\infty, \beta] \to [-\beta, +\infty)$ defined by $T(t) := -t$. Since we know from Thm. 5.1(a) that the measure $\tau \in \mathcal{M}_{\geq}^q([-\beta, +\infty))$ with (9.2) for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$ is unique, we obtain $\tilde{\theta} = \tau$. Hence, σ is the image measure of τ under the transformation R.

(b) Let $\sigma \in \mathcal{M}_{\geq}^q((-\infty, \beta])$ be such that (9.1) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Thus, in view of Prop. A.5, the function $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ fulfills (9.3) for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, where $\tilde{\theta}$ is the image measure of σ under the transformation T. Thm. 5.1(b) yields then $F \in \mathcal{S}_{0,q;[-\beta,+\infty)}$. Because of $G(z) = -[F(-\overline{z})]^*$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, the function G belongs to $\mathcal{S}_{0,q;(-\infty,\beta]}$ by virtue of Rem. 9.1(a). $\qquad\square$

If $\beta \in \mathbb{R}$ and σ is a measure belonging to $\mathcal{M}^q_{\geq}((-\infty,\beta])$, then we will call the matrix-valued function $G: \mathbb{C} \setminus [\alpha,+\infty) \to \mathbb{C}^{q \times q}$ which is, for each $z \in \mathbb{C} \setminus (-\infty,\beta]$, given by (9.1) the $(-\infty,\beta]$-*Stieltjes transform of* σ. If $G \in \mathcal{S}_{0,q;(-\infty,\beta]}$, then the unique measure $\sigma \in \mathcal{M}^q_{\geq}((-\infty,\beta])$ which fulfills (9.1) for each $z \in \mathbb{C} \setminus (-\infty,\beta]$ is said to be the $(-\infty,\beta]$-*Stieltjes measure of* G and will be denoted by σ_G.

Remark 9.3. (a) Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{0,q;[\alpha,+\infty)}$. In view of Thm. 5.1 and Prop. A.5, we have then

$$-[F(-\overline{z})]^* = \int_{[\alpha,+\infty)} \frac{1}{-t-z} \sigma_F(\mathrm{d}t) = \int_{(-\infty,-\alpha]} \frac{1}{t-z} \hat{\theta}(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus (-\infty,-\alpha]$, where $\hat{\theta}$ is the image measure of σ_F under the transformation $R: [\alpha,+\infty) \to (-\infty,-\alpha]$ defined by $R(t) := -t$. Thm. 9.2 yields then that $G: \mathbb{C} \setminus (-\infty,-\alpha] \to \mathbb{C}^{q \times q}$ defined by $G(z) := -[F(-\overline{z})]^*$ belongs to $\mathcal{S}_{0,q;(-\infty,-\alpha]}$ and that $\sigma_G = \hat{\theta}$.

(b) Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{0,q;(-\infty,\beta]}$. In view of Thm. 9.2 and Prop. A.5, we have then

$$-[G(-\overline{z})]^* = \int_{(-\infty,\beta]} \frac{1}{-t-z} \sigma_G(\mathrm{d}t) = \int_{[-\beta,+\infty)} \frac{1}{t-z} \tilde{\theta}(\mathrm{d}t)$$

for all $z \in \mathbb{C} \setminus [-\beta,+\infty)$, where $\tilde{\theta}$ is the image measure of σ_G under the transformation $T: (-\infty,\beta] \to [-\beta,+\infty)$ defined by $T(t) := -t$. Thm. 5.1 yields then that $F: \mathbb{C} \setminus [-\beta,+\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{0,q;[-\beta,+\infty)}$ and that $\sigma_F = \tilde{\theta}$.

Proposition 9.4. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{0,q;(-\infty,\beta]}$. Then $\mathcal{R}(G(z)) = \mathcal{R}(\sigma_G((-\infty,\beta]))$ and $\mathcal{N}(G(z)) = \mathcal{N}(\sigma_G((-\infty,\beta]))$ for all $z \in \mathbb{C} \setminus (-\infty,\beta]$.*

Proof. According to Rem. 9.3(b), the function $F: \mathbb{C} \setminus [-\beta,+\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{0,q;[-\beta,+\infty)}$ and σ_F is the image measure of σ_G under the transformation $T: (-\infty,\beta] \to [-\beta,+\infty)$ defined by $T(t) := -t$. In particular, $\sigma_F([-\beta,+\infty)) = \sigma_G((-\infty,\beta])$. Prop. 5.3 yields $\mathcal{R}(F(w)) = \mathcal{R}(\sigma_F([-\beta,+\infty)))$ and $\mathcal{N}(F(w)) = \mathcal{N}(\sigma_F([-\beta,+\infty)))$ for all $w \in \mathbb{C} \setminus [-\beta,+\infty)$. Furthermore, $\mathcal{R}([F(w)]^*) = \mathcal{R}(F(w))$ and $\mathcal{N}([F(w)]^*) = \mathcal{N}(F(w))$ follow for all $w \in \mathbb{C} \setminus [-\beta,+\infty)$ from Prop. 3.15(b). Finally for all $z \in \mathbb{C} \setminus (-\infty,\beta]$, we infer

$$\mathcal{R}(G(z)) = \mathcal{R}([F(-\overline{z})]^*) = \mathcal{R}(F(-\overline{z})) = \mathcal{R}(\sigma_F([-\beta,+\infty))) = \mathcal{R}(\sigma_G((-\infty,\beta]))$$

and

$$\mathcal{N}(G(z)) = \mathcal{N}([F(-\overline{z})]^*) = \mathcal{N}(F(-\overline{z})) = \mathcal{N}(\sigma_F([-\beta,+\infty))) = \mathcal{N}(\sigma_G((-\infty,\beta])). \quad \square$$

10. Moore-Penrose inverses of functions belonging to the class $\mathcal{S}_{q;(-\infty,\beta]}$

This section is the dual counterpart to Section 6.

Proposition 10.1. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}$. Then $G^{\dagger}$ is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$.*

Proof. In view of Prop. 7.8(b), we obtain the identities $\mathcal{N}(G(z)) = \mathcal{N}(G(\mathrm{i}))$ and $\mathcal{R}(G(z)) = \mathcal{R}(G(\mathrm{i}))$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Thus, the application of [15, Prop. 8.4] completes the proof. $\qquad\square$

Let $\beta \in \mathbb{R}$ and $G \in \mathcal{S}_{q;(-\infty,\beta]}$. Then Lem. 10.1 suggests to look if there are functions closely related to $G^{\dagger}$ which belong again to $\mathcal{S}_{q;(-\infty,\beta]}$. Against to this background, we are led to the function $F \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $F(z) := -(\beta - z)^{-1}[G(z)]^{\dagger}$.

Theorem 10.2. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}$. Then $F \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $F(z) := -(\beta - z)^{-1}[G(z)]^{\dagger}$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}$.*

Proof. According to Rem. 7.1, the function $P \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $P(z) := -[G(-\overline{z})]^{*}$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$. Thm. 6.3 yields then that the function $Q \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $Q(z) := -(z + \beta)^{-1}[P(z)]^{\dagger}$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ as well. We now infer

$$- [Q(-\overline{z})]^{*} = -(-z + \beta)^{-1}\left(-[P(-\overline{z})]^{\dagger}\right)^{*} = -(\beta - z)^{-1}(-[P(-\overline{z})]^{*})^{\dagger} = F(z) \quad (10.1)$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Hence, $F \in \mathcal{S}_{q;(-\infty,\beta]}$ by virtue of Rem. 7.1. $\qquad\square$

Now we specify the result of Thm. 10.2 for functions belonging to $\mathcal{S}_{0,q;(-\infty,\beta]}$.

Proposition 10.3. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{0,q;(-\infty,\beta]}$. Then $F \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $F(z) := -(\beta - z)^{-1}[G(z)]^{\dagger}$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ and $\gamma_{F} = [\sigma_{G}((-\infty,\beta])]^{\dagger}$. If G is not the constant function with value $O_{q \times q}$, then $F \in \mathcal{S}_{q;(-\infty,\beta]} \setminus \mathcal{S}_{0,q;(-\infty,\beta]}$.*

Proof. According to Rem. 9.3(b), the function $P \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $P(z) := -[G(-\overline{z})]^{*}$ belongs to $\mathcal{S}_{0,q;[-\beta,+\infty)}$ and σ_{P} is the image measure of σ_{G} under the transformation $T \colon (-\infty, \beta] \to [-\beta, +\infty)$ defined by $T(t) := -t$. In particular, $\sigma_{P}([-\beta, +\infty)) = \sigma_{G}((-\infty, \beta])$. Prop. 6.4 yields that the function $Q \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $Q(z) := -(z + \beta)^{-1}[P(z)]^{\dagger}$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$, that $\gamma_{Q} = [\sigma_{P}([-\beta, +\infty))]^{\dagger}$, and that $Q \notin \mathcal{S}_{0,q;[-\beta,+\infty)}$ if P is not the constant function with value $O_{q \times q}$. Furthermore, we have (10.1) for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Hence, $F \in \mathcal{S}_{q;(-\infty,\beta]}$ and $\gamma_{F} = \gamma_{Q}$ by virtue of Rem. 7.7(a). We now infer $\gamma_{F} = [\sigma_{G}((-\infty,\beta])]^{\dagger}$. From (10.1) we conclude $Q(z) = -[F(-\overline{z})]^{*}$ for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$. Since $G \not\equiv O_{q \times q}$ implies $P \not\equiv O_{q \times q}$, the proof is complete in view of Rem. 9.1(b). $\qquad\square$

Now we introduce the dual counterpart of the class introduced in Notation 6.5.

Notation 10.4. Let $\beta \in \mathbb{R}$. Then let $\mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$ be the class of all matrix-valued functions $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ which fulfill the following two conditions:

(I) G is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$ with $\mathrm{Rstr}_{\Pi_{+}} G \in \mathcal{R}_{q}(\Pi_{+})$.

(II) For all $x \in (\beta, +\infty)$, the matrix $G(x)$ is non-negative Hermitian.

For the special case $q = 1$ and $\beta = 0$ the class introduced in Notation 10.4 was studied by Katsnelson [18].

Example 10.5. Let $\beta \in \mathbb{R}$ and let $D, E \in \mathbb{C}^{q \times q}_{\geq}$. Then $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $G(z) := D - (\beta - z)E$ belongs to $\mathcal{S}^{[-1]}_{q;(-\infty,\beta]}$.

Remark 10.6. Let $\alpha, \beta \in \mathbb{R}$. Then:

(a) If $F \in \mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$, then $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $G(z) := -[F(\alpha + \beta - \overline{z})]^*$ belongs to $\mathcal{S}^{[-1]}_{q;(-\infty,\beta]}$.

(b) If $G \in \mathcal{S}^{[-1]}_{q;(-\infty,\beta]}$, then $F \colon \mathbb{C} \setminus [\alpha, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(\alpha + \beta - \overline{z})]^*$ belongs to $\mathcal{S}^{[-1]}_{q;[\alpha,+\infty)}$.

Theorem 10.7. *Let $\beta \in \mathbb{R}$ and let $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$. Then:*

(a) *If $G \in \mathcal{S}^{[-1]}_{q;(-\infty,\beta]}$, then there are unique non-negative Hermitian complex $q \times q$ matrices D and E and a unique non-negative Hermitian measure $\rho \in \mathcal{M}^q_{\geq}((-\infty, \beta))$ such that*

$$G(z) = D + (\beta - z)\left[-E + \int_{(-\infty,\beta)} \frac{1 + \beta - t}{t - z} \rho(\mathrm{d}t)\right] \tag{10.2}$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Furthermore, the function $Q \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $Q(z) := (\beta - z)^{-1}G(z)$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}$ with $D = \mu_Q(\{\beta\})$ and $(E, \rho) = (\gamma_Q, \mathrm{Rstr}_{\mathfrak{B}_{(-\infty,\beta)}} \mu_Q)$.

(b) *If $D \in \mathbb{C}^{q \times q}_{\geq}$, $E \in \mathbb{C}^{q \times q}_{\geq}$ and $\rho \in \mathcal{M}^q_{\geq}((-\infty, \beta))$ are such that G can be represented via (10.2) for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, then G belongs to $\mathcal{S}^{[-1]}_{q;(-\infty,\beta]}$.*

Proof. (a) Let $G \in \mathcal{S}^{[-1]}_{q;(-\infty,\beta]}$. According to Rem. 10.6(b), the function $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}^{[-1]}_{q;[-\beta,+\infty)}$. Thm. 6.10(a) yields then that there is a unique triple (M, N, ω) from $\mathbb{C}^{q \times q}_{\geq} \times \mathbb{C}^{q \times q}_{\geq} \times \mathcal{M}^q_{\geq}((-\beta, +\infty))$ such that $F(z) = -M + (z + \beta)[N + \int_{(-\beta,+\infty)}(1 + t + \beta)/(t - z)\omega(\mathrm{d}t)]$ for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$ and that the function $P \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $P(z) := (z + \beta)^{-1}F(z)$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ with $M = \mu_P(\{-\beta\})$ and $(N, \omega) = (\gamma_P, \mathrm{Rstr}_{\mathfrak{B}_{(-\beta,+\infty)}} \mu_P)$. Applying Prop. A.5, we now infer

$$
\begin{aligned}
G(z) = -[F(-\overline{z})]^* &= M - (-z + \beta)\left[N + \int_{(-\beta,+\infty)} \frac{1 + t + \beta}{t + z}\omega(\mathrm{d}t)\right] \\
&= M + (\beta - z)\left[-N + \int_{(-\beta,+\infty)} \frac{1 + \beta - (-t)}{-t - z}\omega(\mathrm{d}t)\right] \\
&= D + (\beta - z)\left[-E + \int_{(-\infty,\beta)} \frac{1 + \beta - t}{t - z}\rho(\mathrm{d}t)\right]
\end{aligned}
$$

for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, where $D := M$, where $E := N$, and where ρ is the image measure of ω under the transformation $R_0: (-\beta, +\infty) \to (-\infty, \beta)$ defined by $R_0(t) := -t$. The function Q fulfills $Q(z) = -[P(-\overline{z})]^*$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Rem. 7.7(a) yields then $Q \in \mathcal{S}_{q;(-\infty,\beta]}$ with $\gamma_Q = \gamma_P$ and μ_Q being the image measure of μ_P under the transformation $R: [-\beta, +\infty) \to (-\infty, \beta]$ defined by $R(t) := -t$. Hence, $D = M = \mu_P(\{-\beta\}) = \mu_Q(\{\beta\})$ and $E = N = \gamma_P = \gamma_Q$ follow. Furthermore, $\mathrm{Rstr}_{\mathfrak{B}_{(-\infty,\beta)}} \mu_Q$ is the image measure of $\mathrm{Rstr}_{\mathfrak{B}_{(-\beta,+\infty)}} \mu_P$ under the transformation R_0, implying $\rho = \mathrm{Rstr}_{\mathfrak{B}_{(-\infty,\beta)}} \mu_Q$. In particular, the triple (D, E, ρ) is unique.

(b) Let $D, E \in \mathbb{C}_{\succeq}^{q \times q}$ and let $\rho \in \mathcal{M}_{\succeq}^q((-\infty, \beta))$ be such that (10.2) holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Let $F: \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ be defined by $F(z) := -[G(-\overline{z})]^*$. Then, using Prop. A.5, we get

$$
\begin{aligned}
F(z) = -[G(-\overline{z})]^* &= -D - (\beta + z)\left[-E + \int_{(-\infty,\beta)} \frac{1 + \beta - t}{t + z} \rho(\mathrm{d}t)\right] \\
&= -D + (z + \beta)\left[E + \int_{(-\infty,\beta)} \frac{1 - t + \beta}{-t - z} \rho(\mathrm{d}t)\right] \\
&= -D + (z + \beta)\left[E + \int_{(-\beta,\infty)} \frac{1 + t + \beta}{t - z} \tilde{\theta}(\mathrm{d}t)\right]
\end{aligned}
$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, where $\tilde{\theta}$ is the image measure of ρ under the transformation $T_0: (-\infty, \beta) \to (-\beta, +\infty)$ defined by $T_0(t) := -t$. Thm. 6.10(b) yields then $F \in \mathcal{S}_{q;[-\beta,+\infty)}^{[-1]}$. In view of $G(z) = -[F(-\overline{z})]^*$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, hence G belongs to $\mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$ by virtue of Rem. 10.6(a). $\qquad\square$

In the following, if $\beta \in \mathbb{R}$ and $G \in \mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$ are given, then we will write (D_G, E_G, ρ_G) for the unique triple $(D, E, \rho) \in \mathbb{C}_{\succeq}^{q \times q} \times \mathbb{C}_{\succeq}^{q \times q} \times \mathcal{M}_{\succeq}^q((-\infty, \beta))$ which fulfills (10.2) for all $z \in \mathbb{C} \setminus (-\infty, \beta]$.

Remark 10.8. (a) Let $\alpha \in \mathbb{R}$ and let $F \in \mathcal{S}_{q;[\alpha,+\infty)}^{[-1]}$. In view of Thm. 6.10(a) and Prop. A.5, we have then

$$
\begin{aligned}
-[F(-\overline{z})]^* &= D_F - (-z - \alpha)\left[E_F + \int_{(\alpha,+\infty)} \frac{1 + t - \alpha}{t + z} \rho_F(\mathrm{d}t)\right] \\
&= D_F + (-\alpha - z)\left[-E_F + \int_{(\alpha,+\infty)} \frac{1 - \alpha - (-t)}{-t - z} \rho_F(\mathrm{d}t)\right] \\
&= D_F + (-\alpha - z)\left[-E_F + \int_{(-\infty,-\alpha)} \frac{1 - \alpha - t}{t - z} \hat{\theta}(\mathrm{d}t)\right]
\end{aligned}
$$

for all $z \in \mathbb{C} \setminus (-\infty, -\alpha]$, where $\hat{\theta}$ is the image measure of ρ_F under the transformation $R_0: (\alpha, +\infty) \to (-\infty, -\alpha)$ defined by $R_0(t) := -t$. Because of $D_F \in \mathbb{C}_{\succeq}^{q \times q}$

and $E_F \in \mathbb{C}_{\geq}^{q \times q}$, Thm. 10.7 yields then that $G \colon \mathbb{C} \setminus (-\infty, -\alpha] \to \mathbb{C}^{q \times q}$ defined by $G(z) := -[F(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;(-\infty,-\alpha]}^{[-1]}$ and that $(D_G, E_G, \rho_G) = (D_F, E_F, \hat{\theta})$.

(b) Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$. In view of Thm. 10.7(a) and Prop. A.5, we have then

$$
\begin{aligned}
-[G(-\overline{z})]^* &= -D_G - (\beta + z)\left[-E_G + \int_{(-\infty,\beta)} \frac{1 + \beta - t}{t + z} \rho_G(\mathrm{d}t) \right] \\
&= -D_G + (z + \beta)\left[E_G + \int_{(-\infty,\beta)} \frac{1 - t + \beta}{-t - z} \rho_G(\mathrm{d}t) \right] \\
&= -D_G + (z + \beta)\left[E_G + \int_{(-\beta,+\infty)} \frac{1 + t + \beta}{t - z} \tilde{\theta}(\mathrm{d}t) \right]
\end{aligned}
$$

for all $z \in \mathbb{C} \setminus [-\beta, +\infty)$, where $\tilde{\theta}$ is the image measure of ρ_G under the transformation $T_0 \colon (-\infty, \beta) \to (-\beta, +\infty)$ defined by $T_0(t) := -t$. Because of $D_G \in \mathbb{C}_{\geq}^{q \times q}$ and $E_G \in \mathbb{C}_{\geq}^{q \times q}$, Thm. 6.10 yields then that $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}^{[-1]}$ and that $(D_F, E_F, \rho_F) = (D_G, E_G, \tilde{\theta})$.

Corollary 10.9. *Let $\beta \in \mathbb{R}$. Then $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$ if and only if $Q \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $Q(z) = (\beta - z)^{-1} G(z)$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}$.*

Proof. From Rem. 10.8 we conclude that G belongs to $\mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$ if and only if $F \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}^{[-1]}$. Cor. 6.11 yields that F belongs to $\mathcal{S}_{q;[-\beta,+\infty)}^{[-1]}$ if and only if $P \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $P(z) = (z + \beta)^{-1} F(z)$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$. Since $Q(z) = -[P(-\overline{z})]^*$ holds true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, we see by virtue of Rem. 7.7 that P belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ if and only if Q belongs to $\mathcal{S}_{q;(-\infty,\beta]}$, which completes the proof. $\square$

Corollary 10.10. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$. For all $x_1, x_2 \in (\beta, +\infty)$ with $x_1 \leq x_2$, then $O_{q \times q} \leq G(x_1) \leq G(x_2)$.*

Proof. Using Thm. 10.7, we obtain $G(x_2) - G(x_1) = (x_2 - x_1)(E_G + \int_{(-\infty,\beta]}[(1 + \beta - t)(\beta - t)]/[(t - x_2)(t - x_1)]\rho_G(\mathrm{d}t))$ for all $x_1, x_2 \in (\beta, +\infty)$ with $x_1 \leq x_2$, by direct calculation. Since $E_G \in \mathbb{C}_{\geq}^{q \times q}$ and $G(x) \in \mathbb{C}_{\geq}^{q \times q}$ for all $x \in (\beta, +\infty)$, thus the proof is complete. $\square$

Proposition 10.11. *Let $\beta \in \mathbb{R}$ and let $G \in \mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$. Then:*

(a) *If $z \in \mathbb{C} \setminus (-\infty, \beta]$, then $[G(z)]^* = G(\overline{z})$.*

(b) *For all $z \in \mathbb{C} \setminus (-\infty, \beta]$,*

$$\mathcal{R}(G(z)) = \mathcal{R}(D_G) + \mathcal{R}(E_G) + \mathcal{R}(\rho_G((-\infty, \beta))), \tag{10.3}$$

$$\mathcal{N}(G(z)) = \mathcal{N}(D_G) \cap \mathcal{N}(E_G) \cap \mathcal{N}(\rho_G((-\infty, \beta))), \tag{10.4}$$

and, in particular, $\mathcal{R}([G(z)]^*) = \mathcal{R}(G(z))$ and $\mathcal{N}([G(z)]^*) = \mathcal{N}(G(z))$.

(c) Let $r \in \mathbb{N}_0$. Then the following statements are equivalent:

(i) $\operatorname{rank} G(z) = r$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$.

(ii) There is some $z_0 \in \mathbb{C} \setminus (-\infty, \beta]$ such that $\operatorname{rank} G(z_0) = r$.

(iii) $\dim[\mathcal{R}(D_G) + \mathcal{R}(E_G) + \mathcal{R}(\rho_G((-\infty, \beta)))] = r$.

Proof. (a) This can be concluded from the representation (10.2) in Thm. 10.7(a).

(b) According to Rem. 10.8(b), the function $F: \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}^{[-1]}$ and $(D_F, E_F, \rho_F) = (D_G, E_G, \tilde{\theta})$, where $\tilde{\theta}$ is the image measure of ρ_G under the transformation $T_0: (-\infty, \beta) \to (-\beta, +\infty)$ defined by $T_0(t) := -t$. In particular, $\rho_F((-\beta, +\infty)) = \rho_G((-\infty, \beta))$. Prop. 6.14(b) yields $\mathcal{R}([F(w)]^*) = \mathcal{R}(F(w)) = \mathcal{R}(D_F) + \mathcal{R}(E_F) + \mathcal{R}(\rho_F((-\beta, +\infty)))$ and $\mathcal{N}([F(w)]^*) = \mathcal{N}(F(w)) = \mathcal{N}(D_F) \cap \mathcal{N}(E_F) \cap \mathcal{N}(\rho_F((-\beta, +\infty)))$ for all $w \in \mathbb{C} \setminus [-\beta, +\infty)$. We infer

$$\mathcal{R}(G(z)) = \mathcal{R}([F(-\overline{z})]^*) = \mathcal{R}(D_F) + \mathcal{R}(E_F) + \mathcal{R}(\rho_F((-\beta, +\infty)))$$
$$= \mathcal{R}(D_G) + \mathcal{R}(E_G) + \mathcal{R}(\rho_G((-\infty, \beta)))$$

and, analogously, $\mathcal{N}(G(z)) = \mathcal{N}(D_G) \cap \mathcal{N}(E_G) \cap \mathcal{N}(\rho_G((-\infty, \beta)))$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. In view of (a), part (b) is proved.

(c) This is an immediate consequence of (b). $\qquad\square$

Corollary 10.12. *Let $\beta \in \mathbb{R}$, let $G \in \mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$, and let $z_0 \in \mathbb{C} \setminus (-\infty, \beta]$. Then $G(z_0) = O_{q \times q}$ if and only if $G(z) = O_{q \times q}$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$.*

Proof. This is an immediate consequence of Prop. 10.11(c). $\qquad\square$

Corollary 10.13. *Let $\beta \in \mathbb{R}$, let $G \in \mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$, and let $\lambda \in \mathbb{R}$ be such that the matrix $D_G - \lambda I_q$ is non-negative Hermitian. Then*

$$\mathcal{R}(G(z) - \lambda I_q) = \mathcal{R}(G(w) - \lambda I_q) \quad and \quad \mathcal{N}(G(z) - \lambda I_q) = \mathcal{N}(G(w) - \lambda I_q) \quad (10.5)$$

for every choice of z and w in $\mathbb{C} \setminus (-\infty, \beta]$. In particular, if $\lambda \leq 0$, then λ is an eigenvalue of the matrix $G(z_0)$ for some $z_0 \in \mathbb{C} \setminus (-\infty, \beta]$ if and only if λ is an eigenvalue of the matrix $G(z)$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. In this case, the eigenspaces $\mathcal{N}(G(z) - \lambda I_q)$ are independent of $z \in \mathbb{C} \setminus (-\infty, \beta]$.

Proof. In view of Thm. 10.7, we conclude that the function $F: \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ defined by $F(z) := G(z) - \lambda I_q$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$. The application of Prop. 10.11(b) to the function F yields then (10.5). Since the matrix D_G is non-negative Hermitian, we have $D_G - \lambda I_q \in \mathbb{C}_{\geq}^{q \times q}$ if $\lambda \leq 0$. Thus, the remaining assertions are an immediate consequence of (10.5). $\qquad\square$

Lemma 10.14. *Let $\beta \in \mathbb{R}$ and $G \in \mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$. Then $G^\dagger$ is holomorphic in $\mathbb{C} \setminus (-\infty, \beta]$.*

Proof. In view of formulas (10.3) and (10.4), we obtain $\mathcal{N}(G(z)) = \mathcal{N}(G(\mathrm{i}))$ and $\mathcal{R}(G(z)) = \mathcal{R}(G(\mathrm{i}))$ for all $z \in \mathbb{C} \setminus (-\infty, \beta]$. Thus, the application of [15, Prop. 8.4] completes the proof. $\qquad\square$

The following result is an analogue of Thm. 6.18.

Theorem 10.15. *Let $\beta \in \mathbb{R}$ and $G \colon \mathbb{C} \setminus (-\infty, \beta] \to \mathbb{C}^{q \times q}$ be a matrix-valued function. Then G belongs to $\mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$ if and only if $F := -G^{\dagger}$ belongs to $\mathcal{S}_{q;(-\infty,\beta]}$.*

Proof. From Rem. 10.8 we get that G belongs to $\mathcal{S}_{q;(-\infty,\beta]}^{[-1]}$ if and only if $Q \colon \mathbb{C} \setminus [-\beta, +\infty) \to \mathbb{C}^{q \times q}$ defined by $Q(z) := -[G(-\overline{z})]^*$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}^{[-1]}$. Thm. 6.18 yields that Q belongs to $\mathcal{S}_{q;[-\beta,+\infty)}^{[-1]}$ if and only if $P := -Q^{\dagger}$ belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$. Since $F(z) = -[P(-\overline{z})]^*$ is true for all $z \in \mathbb{C} \setminus (-\infty, \beta]$, we see from Rem. 7.7 that P belongs to $\mathcal{S}_{q;[-\beta,+\infty)}$ if and only if F belongs to $\mathcal{S}_{q;(-\infty,\beta]}$, which completes the proof. $\qquad\square$

A. Some considerations on non-negative Hermitian measures

In this appendix, we summarize some facts on integration with respect to non-negative Hermitian measures. For each non-negative Hermitian $q \times q$ measure $\mu = (\mu_{jk})_{j,k=1}^{q}$ on a measurable space $(\Omega, \mathfrak{A})$, we denote by $\mathcal{L}^{1}(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ the set of all $\mathfrak{A}\text{-}\mathfrak{B}_{\mathbb{C}}$-measurable functions $f \colon \Omega \to \mathbb{C}$ such that the integral $\int_{\Omega} f \mathrm{d}\mu$ exists, i.e. that $\int_{\Omega} |f| \mathrm{d}\tilde{\mu}_{jk} < \infty$ for every choice of $j, k \in \mathbb{Z}_{1,q}$, where $\tilde{\mu}_{jk}$ is the variation of the complex measure μ_{jk}.

For each $A \in \mathbb{C}^{q \times q}$, let $\operatorname{tr} A$ be the trace of A.

Remark A.1. Let μ be a non-negative Hermitian measure on a measurable space $(\Omega, \mathfrak{A})$, let $\tau := \operatorname{tr} \mu$ be the trace measure of μ, and let $f \colon \Omega \to \mathbb{C}$ be an $\mathfrak{A}\text{-}\mathfrak{B}_{\mathbb{C}}$-measurable function. Then f belongs to $\mathcal{L}^{1}(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ if and only if f belongs to $\mathcal{L}^{1}(\Omega, \mathfrak{A}, \tau; \mathbb{C})$.

Remark A.2. Let μ be a non-negative Hermitian $q \times q$ measure on a measurable space $(\Omega, \mathfrak{A})$ and let $u \in \mathbb{C}^{q}$. Then $\nu := u^* \mu u$ is a finite measure on $(\Omega, \mathfrak{A})$ which is absolutely continuous with respect to the trace measure of μ. If f belongs to $\mathcal{L}^{1}(\Omega, \mathfrak{A}, \mu; \mathbb{C})$, then $\int_{\Omega} |f| \mathrm{d}\nu < \infty$ and $\int_{\Omega} f \mathrm{d}\nu = u^*(\int_{\Omega} f \mathrm{d}\mu)u$.

Remark A.3. Let μ be a non-negative Hermitian $q \times q$ measure on a measurable space $(\Omega, \mathfrak{A})$. An $\mathfrak{A}\text{-}\mathfrak{B}_{\mathbb{C}}$-measurable function $f \colon \Omega \to \mathbb{C}$ belongs to $\mathcal{L}^{1}(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ if and only if $\int_{\Omega} |f| \mathrm{d}(u^* \mu u) < \infty$ for all $u \in \mathbb{C}^{q}$.

Lemma A.4 (cf. [11, Lem. B.2]). *Let $(\Omega, \mathfrak{A})$ be a measurable space, let σ be a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$, and let τ be the trace measure of σ. Then:*

(a) *If $f \in \mathcal{L}^{1}(\Omega, \mathfrak{A}, \sigma; \mathbb{C})$, then $\mathcal{R}(\int_{\Omega} f \mathrm{d}\sigma) \subseteq \mathcal{R}(\sigma(\Omega))$ and $\mathcal{N}(\sigma(\Omega)) \subseteq \mathcal{N}(\int_{\Omega} f \mathrm{d}\sigma)$.*

(b) *If $f \in \mathcal{L}^{1}(\Omega, \mathfrak{A}, \sigma; \mathbb{C})$ fulfills $\tau(\{f \notin (0, +\infty)\}) = 0$, then $\mathcal{R}(\int_{\Omega} f \mathrm{d}\sigma) = \mathcal{R}(\sigma(\Omega))$ and $\mathcal{N}(\sigma(\Omega)) = \mathcal{N}(\int_{\Omega} f \mathrm{d}\sigma)$.*

A. Some considerations on non-negative Hermitian measures

Proposition A.5 ([12, Prop. B.1]). *Let $(\Omega, \mathfrak{A})$ and $(\tilde{\Omega}, \tilde{\mathfrak{A}})$ be measurable spaces and let μ be a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$. Further, let $T : \Omega \to \tilde{\Omega}$ be an $\mathfrak{A}$-$\tilde{\mathfrak{A}}$-measurable mapping. Then $T(\mu) : \tilde{\mathfrak{A}} \to \mathbb{C}^{q \times q}$ defined by $[T(\mu)](\tilde{A}) := \mu(T^{-1}(\tilde{A}))$ is a non-negative Hermitian $q \times q$ measure on $(\tilde{\Omega}, \tilde{\mathfrak{A}})$. Furthermore, if $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ is an $\tilde{\mathfrak{A}}$-$\mathfrak{B}_{\mathbb{C}}$-measurable mapping, then $\tilde{f} \in \mathcal{L}^1(\tilde{\Omega}, \tilde{\mathfrak{A}}, T(\mu); \mathbb{C})$ if and only if $\tilde{f} \circ T \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$. If $\tilde{f}$ belongs to $\mathcal{L}^1(\tilde{\Omega}, \tilde{\mathfrak{A}}, T(\mu); \mathbb{C})$, then $\int_{\tilde{A}} \tilde{f} \mathrm{d}[T(\mu)] = \int_{T^{-1}(\tilde{A})} (\tilde{f} \circ T) \mathrm{d}\mu$ for all $\tilde{A} \in \tilde{\mathfrak{A}}$.*

Proposition A.6 (Lebesgue's dominated convergence for non-negative Hermitian measures). *Let μ be a non-negative Hermitian $q \times q$ measure on a measurable space $(\Omega, \mathfrak{A})$ with trace measure τ. For all $n \in \mathbb{N}$, let $f_n : \Omega \to \mathbb{C}$ be an $\mathfrak{A}$-$\mathfrak{B}_{\mathbb{C}}$-measurable function. Let $f : \Omega \to \mathbb{C}$ be an $\mathfrak{A}$-$\mathfrak{B}_{\mathbb{C}}$-measurable function and let $g \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ be such that $\lim_{n \to +\infty} f_n(\omega) = f(\omega)$ for τ-a. a. $\omega \in \Omega$ and that $|f_n(\omega)| \leq |g(\omega)|$ for all $n \in \mathbb{N}$ and τ-a. a. $\omega \in \Omega$. Then $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$, $f_n \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ for all $n \in \mathbb{N}$, and $\lim_{n \to +\infty} \int_\Omega f_n \mathrm{d}\mu = \int_\Omega f \mathrm{d}\mu$.*

Proof. Let $u \in \mathbb{C}^q$ and let $\nu := u^* \mu u$. According to Rem. A.2, then ν is a finite measure on $(\Omega, \mathfrak{A})$ which is absolutely continuous with respect to τ and $\int_\Omega |g| \mathrm{d}\nu < \infty$. In particular, $\lim_{n \to +\infty} f_n(\omega) = f(\omega)$ for ν-a. a. $\omega \in \Omega$ and $|f_n(\omega)| \leq |g(\omega)|$ for all $n \in \mathbb{N}$ and ν-a. a. $\omega \in \Omega$. Thus, Lebesgue's dominated convergence theorem provides us $\int_\Omega |f| \mathrm{d}\nu < \infty$, $\int_\Omega |f_n| \mathrm{d}\nu < \infty$ for all $n \in \mathbb{N}$, and $\lim_{n \to +\infty} \int_\Omega f_n \mathrm{d}\nu = \int_\Omega f \mathrm{d}\nu$. Since $u \in \mathbb{C}^q$ was arbitrarily chosen, $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ and $f_n \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ for all $n \in \mathbb{N}$ follow by virtue of Rem. A.3. Thus, using Rem. A.2, we get $\lim_{n \to +\infty} u^* (\int_\Omega f_n \mathrm{d}\mu) u = u^* (\int_\Omega f \mathrm{d}\mu) u$ for all $u \in \mathbb{C}^q$. Hence, $\lim_{n \to +\infty} \int_\Omega f_n \mathrm{d}\mu = \int_\Omega f \mathrm{d}\mu$ holds true. $\qquad\square$

Remark A.7. If $A \in \mathbb{C}^{q \times q}$ is such that $\mathrm{Im}\, A \in \mathbb{C}_{\geq}^{q \times q}$, then $\mathcal{N}(A) \subseteq \mathcal{N}(\mathrm{Im}\, A)$ (see, e. g. [10, Lem. A.10]).

Lemma A.8. *Let Ω be a non-empty closed subset of $\mathbb{R}$, let $\sigma \in \mathcal{M}_{\geq}^q(\Omega)$, and let τ be the trace measure of σ. Then:*

   (a) For each $w \in \mathbb{C} \setminus \Omega$, the function $g_w : \Omega \to \mathbb{C}$ defined by $g_w(t) := 1/(t - w)$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$.

   (b) The matrix-valued function $S : \mathbb{C} \setminus \Omega \to \mathbb{C}^{q \times q}$ given by $S(w) := \int_\Omega g_w \mathrm{d}\sigma$ satisfies $\mathcal{R}(S(z)) = \mathcal{R}(\sigma(\Omega))$ and, in particular, $\mathrm{rank}\, S(z) = \mathrm{rank}\, \sigma(\Omega)$ for each $z \in \mathbb{C} \setminus [\inf \Omega, \sup \Omega]$.

   (c) $-\mathrm{i} \lim_{y \to +\infty} y S(\mathrm{i}y) = \sigma(\Omega)$.

Proof. Part (a) is readily checked. In particular, the function S is well defined. Let $z \in \mathbb{C} \setminus [\inf \Omega, \sup \Omega]$. From Lem. A.4(a) we get then

$$\mathcal{R}(S(z)) = \mathcal{R}\left(\int_\Omega g_z \mathrm{d}\sigma \right) \subseteq \mathcal{R}(\sigma(\Omega)).$$

Thus, in order to prove part (b), it remains to verify that $\mathcal{R}(\sigma(\Omega)) \subseteq \mathcal{R}(S(z))$, i. e., that

$$\mathcal{N}(S^*(z)) \subseteq \mathcal{N}(\sigma(\Omega)) \tag{A.1}$$

is true. Part (a) implies
$$\operatorname{Im} g_z \in \mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C}). \tag{A.2}$$

For each $t \in \Omega$, one can easily check the equations
$$g_z(t) = \frac{t - \overline{z}}{|t - z|^2}, \qquad \overline{g_z}(t) = \frac{t - z}{|t - z|^2}, \qquad \text{and} \qquad \operatorname{Im} g_z(t) = \frac{\operatorname{Im} z}{|t - z|^2}. \tag{A.3}$$

First we consider the case that z belongs to Π_+. In view of $\operatorname{Im}(-S^*(z)) = \operatorname{Im} S(z) = \int_\Omega \operatorname{Im} g_z \mathrm{d}\sigma$ and (A.3), we see that the matrix $\operatorname{Im}(-S^*(z))$ is non-negative Hermitian. Rem. A.7 and (A.3) yield then

$$\mathcal{N}(S^*(z)) = \mathcal{N}(-S^*(z)) \subseteq \mathcal{N}(\operatorname{Im}(-S^*(z))) = \mathcal{N}\left(\int_\Omega \operatorname{Im} g_z \mathrm{d}\sigma\right). \tag{A.4}$$

From (A.3) we know that $\tau(\{\operatorname{Im} g_z \in (-\infty, 0]\}) = \tau(\emptyset) = 0$. Taking into account (A.2) and Lem. A.4(b), then

$$\mathcal{N}\left(\int_\Omega \operatorname{Im} g_z \mathrm{d}\sigma\right) = \mathcal{N}(\sigma(\Omega)) \tag{A.5}$$

follows. Combining (A.4) and (A.5) we obtain (A.1).

Now we study the case that z belongs to Π_-. By virtue of

$$\operatorname{Im} S^*(z) = -\operatorname{Im} S(z) = \int_\Omega (-\operatorname{Im} g_z) \mathrm{d}\sigma \tag{A.6}$$

and (A.3), the matrix $\operatorname{Im} S^*(z)$ belongs to $\mathbb{C}_{\geq}^{q \times q}$. Hence, Rem. A.7 and (A.6) yield

$$\mathcal{N}(S^*(z)) \subseteq \mathcal{N}(\operatorname{Im} S^*(z)) = \mathcal{N}\left(\int_\Omega (-\operatorname{Im} g_z) \mathrm{d}\sigma\right). \tag{A.7}$$

Because of (A.3), we have $\tau(\{-\operatorname{Im} g_z \in (-\infty, 0]\}) = \tau(\emptyset) = 0$. Thus, using (A.2) and Lem. A.4(b), we have $\mathcal{N}(\int_\Omega (-\operatorname{Im} g_z) \mathrm{d}\sigma) = \mathcal{N}(\sigma(\Omega))$. Taking into account (A.7), this implies (A.1).

Now we discuss the case that $\inf \Omega > -\infty$ and that $z \in (-\infty, \inf \Omega)$. In view of $t \in \Omega$ and (A.3), we have then $\tau(\{\overline{g_z} \in (-\infty, 0]\}) = \tau(\emptyset) = 0$. Since one can see from part (a) that $\overline{g_z}$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$, application of Lem. A.4(b) provides us then

$$\mathcal{N}(S^*(z)) = \mathcal{N}\left(\int_\Omega \overline{g_z} \mathrm{d}\sigma\right) = \mathcal{N}(\sigma(\Omega)).$$

Hence, (A.1) is valid. Similarly, in the case that $\sup \Omega < +\infty$ and $z \in (\sup \Omega, +\infty)$ hold, we get that (A.1) is fulfilled. Thus (A.1) is verified for each $z \in \mathbb{C} \setminus [\inf \Omega, \sup \Omega]$.

(c) For each $y \in (0, +\infty)$ and each $t \in \Omega$, we have $\operatorname{Re}((-\mathrm{i}y)/(t - \mathrm{i}y)) = y^2/(t^2 + y^2)$, $\operatorname{Im}((-\mathrm{i}y)/(t - \mathrm{i}y)) = -yt/(t^2 + y^2)$, and $|(-\mathrm{i}y)/(t - \mathrm{i}y)| \leq 1$. From Prop. A.6 we obtain then

$$\sigma(\Omega) = \int_\Omega 1 \mathrm{d}\sigma = \lim_{y \to +\infty} \int_\Omega \frac{-\mathrm{i}y}{t - \mathrm{i}y} \sigma(\mathrm{d}t) = -\mathrm{i} \lim_{y \to +\infty} y S(\mathrm{i}y). \qquad \square$$

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